ASYMPTOTIC DIMENSION OF GROUPS

By

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Gromov introduced the asymptotic dimension (asdim) of a metric space in his study of asymptotic invariants of infinite groups. Later, Yu introduced a property of discrete metric spaces and groups called property A. Yu showed that for groups with finite asdim or with property A, the famous Novikov Higher Signature Conjecture holds. Recently Gromov and others provided means of constructing groups without either of these properties, so the problem of determining precisely which groups have these properties became interesting. In this dissertation we prove that both of these properties are preserved under the formation of certain infinite unions, amalgamated free products and HNN extensions. In the case of asdim we also give an upper bound which is tight for the HNN extension and many amalgamated free products.

CHAPTER 1
INTRODUCTION

1.1 The Asymptotic Approach

Let Γ be a finitely generated group with a fixed generating set $S = S^{-1}$. There is a left-invariant metric d_S on Γ corresponding to S for which $d_S(s, e) = 1$ for all $s \in S$, where e is the group identity. Thus, we can associate to the pair (Γ, S) the metric space $|\Gamma|_S$ and consider its geometric and topological properties.

The local geometric and topological structure of this metric space is boring, as it is a discrete metric space. In order to gain information about the group Γ from this metric space, one considers what happens to $|\Gamma|_S$ as the "distance to the observer tends to infinity." In this approach we seek to understand phenomena that occur on the R-scale, where R is arbitrarily large, and largely ignore bounded or local details. This is in contrast with the standard calculus, where ϵ-scale phenomena are considered with $\epsilon > 0$ very small.

As an example indicating this philosophy, consider the integer lattice in the plane, i.e., those points (x, y) where x and y are both integers. The local geometry is that of a discrete space and so is not interesting, but if one imagines increasing the distance from the lattice to the observer, then the points of the lattice seem to get closer to one another. As one continues this procedure to infinity, the points seem to coalesce to form a continuous plane without any gaps or holes in it. Thus, from the asymptotic perspective it is impossible to distinguish the integer lattice in the plane from the plane itself. Although it may appear that this "coarsening" has the effect of destroying information, we shall see that for the metric space $|\Gamma|_S$, this procedure yields a great deal of information about the group Γ.

The asymptotic approach (due to Gromov [11]) seeks to capture this coarse structure without appealing to a limiting procedure as above. Instead we consider the metric space $|\Gamma|_S$ as it stands and focus on its *asymptotic properties*, i.e., those properties which remain at the R-scale, with $R > 0$ very large. One of these properties (asdim) is a coarse analog of Lebesgue's covering dimension. We consider this *asymptotic dimension* at length in this paper. As a preliminary example, consider the integer lattice in $\mathbb{R}^n$. As it is a discrete space, its covering dimension is 0, but we have seen that it is "coarsely equivalent" to $\mathbb{R}^n$, so $\text{asdim}\mathbb{Z}^n = \text{asdim}\mathbb{R}^n = n$, (see Dranishnikov-Keesling-Uspenskij [9]).

1.2 Statement of the Main Problem

We again consider the metric space $|\Gamma|_S$ associated to a fixed generating set S of a group Γ. We shall see later that the choice of generating set has no effect on the asdim of the group. Thus, the asdim is a characteristic intrinsic to the group and not the resulting metric space given by a particular choice of generating set. We will therefore abuse notation and refer to $\text{asdim}\Gamma$ instead of $\text{asdim}|\Gamma|_S$.

Guoliang Yu [18] has shown that the finiteness of $\text{asdim}\Gamma$ has important implications to the coarse Baum-Connes and the Novikov Conjectures for the group Γ (see the appendix for the statements of these conjectures). For some time it was unclear whether there were any finitely generated groups with infinite asdim. Both Gromov [12] and Dranishnikov [7] recently gave methods of constructing such an example, and so the question of determining precisely which groups have finite asdim became an interesting one.

Gromov [11] showed that hyperbolic groups have finite asdim. Dranishnikov and Januszkiewicz [8] showed that Coxeter groups have finite asdim. It is clear that finite asdim is preserved by direct products and it can be shown that semi-direct products preserve the finiteness of asdim. The statement of the Novikov Conjecture for a group Γ is really a statement about fundamental groups of certain manifolds, so it is natural to want to estimate the asymptotic dimension of groups which arise as fundamental groups. In particular, one would like to estimate the asymptotic dimension of constructions that arise in connection with the fundamental group.

The free product with amalgamation and the HNN extension both arise in this way (see Chapter 2). We seek an upper bound for the asymptotic dimension of these constructions. Indeed, the question of whether these constructions preserve finite dimensionality or not is not obvious.

Yu [19] introduced a property for discrete metric spaces and groups called property A, which for groups is a weak form of amenability. He showed that if a group has property A then the coarse Baum-Connes and Novikov Conjectures hold for that group. It is known that amenable groups and discrete metric spaces with bounded geometry and finite asdim have property A [13]. It is not difficult to see that the direct and semidirect product of two groups with property A also have property A, but the corresponding question for free products with amalgamation and for HNN extensions was open. Jean-Louis Tu [17] recently showed that these constructions preserve property A by showing that the more general construction of fundamental groups of graphs of groups preserves property A.

Since if a group is finitely generated the corresponding metric space has bounded geometry (see Chapter 2), a finitely generated group with finite asymptotic dimension will have property A. One goal of this paper is to develop a technique for resolving questions about finite dimensionality which are applicable (with minor changes) to the corresponding questions about property A.

1.3 Our Results

In Chapter 2 we recall the basic definitions and elementary results needed in the main body of the paper.

In Chapter 3 we prove that the constructions of free products with amalgamation and HNN extension preserve asymptotic finite dimensionality. Most of these results can be found in Bell-Dranishnikov [3]. Specifically, we estimate the dimension of groups acting in a particular way on finite dimensional metric spaces and combine this estimate with known results on the correspondence between free products with amalgamation and HNN extensions, and actions of groups on trees. We also prove a useful union theorem (Theorem 3.1.1) for asdim which is in stark contrast to union theorems in classical (covering) dimension. Finally, we apply our results to Haefliger's construction of complexes of groups [5] to prove that the fundamental group of a certain type of complex of groups has finite asdim.

In Chapter 3 we are concerned only with proving that the constructions we consider preserve the finiteness of asdim. Because of this, the upper bound estimates given are not optimal. In Chapter 4, we refine the estimates on the upper bound of asdim by passing to the easier case of groups acting on trees (which have asymptotic dimension 1). In particular, we provide the best upper bound for the dimension of an HNN extension and for many amalgamated free products. We prove this upper bound theorem for all fundamental groups of graphs of groups. This construction, due to Bass [1] and Serre [16], is a generalization of the basic constructions of amalgamated free products and HNN extensions. These results can be found in Bell-Dranishnikov [4].

In the final chapter we use the same approach as in Chapters 3 and 4 to prove that groups acting in a certain way on metric spaces with finite asdim have property A, provided certain conditions on the stabilizers hold. These results can be found in Bell [2]. With this and a union theorem also proved in chapter 5, we obtain an alternate proof of Tu's theorem which says that the fundamental group of a graph of groups where each vertex has property A also has property A. In particular, then, the amalgamated product and HNN extension preserve property A. We also prove that fundamental groups of certain complexes of groups have property A, thus generalizing Tu's result.

CHAPTER 2
PRELIMINARIES

2.1 Asymptotic Dimension of Metric Spaces

The notion of the asymptotic dimension, asdim, of a metric space was defined by Gromov [11] to study asymptotic invariants of infinite groups. It is a coarse analog of Ostrand's characterization of covering dimension [14]. He gave three equivalent definitions. We shall use all three versions of the definition in this paper. Let (X, dist) be a metric space.

Asdim(1) Call a family $\mathcal{U}$ of subsets of X d-disjoint if $\mathrm{dist}(U, V) > d$ for every $U \neq V$ in $\mathcal{U}$. Define $\mathrm{asdim} X \leq n$ if for every (large) $d > 0$ there exist $n+1$ uniformly bounded families $\mathcal{U}_0, \dots, \mathcal{U}_n$ of d-disjoint sets covering X.

Asdim(2) Recall that the *order* (or *multiplicity*) of a cover $\mathcal{U}$ of the topological space X is the maximum number n so that every $x \in X$ is contained in no more than n elements of the cover. For any $d > 0$ the *d-multiplicity* of the cover $\mathcal{U}$ of the metric space X is the maximum number n so that every ball of radius d no intersects more than n elements of the cover. Then, $\mathrm{asdim} X \leq n$ if for any (large) $d > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X with d-multiplicity $\leq n+1$.

Asdim(2′) Gromov's second definition can be reformulated in terms of the Lebesgue number of a cover. Recall that the Lebesgue number $L(\mathcal{U})$ of the cover $\mathcal{U}$ of the metric space X is

$$L(\mathcal{U}) = \inf\{\max\{\mathrm{dist}(x, X \setminus U) \mid U \in \mathcal{U}\} x \in X\}.$$

Then, $\mathrm{asdim} X \leq n$ if for any (large) $d > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X with multiplicity $\leq n+1$ and $L(\mathcal{U}) > d$.

Asdim(3) Let K be a simplicial complex. A map $p : X \to K$ is called *uniformly cobounded* if $p^{-1}(\sigma)$ is uniformly bounded for all simplices σ. Let K be a countable simplicial complex, and map K to the Hilbert space l_2 by sending each vertex of K to some element of an orthonormal basis for l_2 and extending the map to all of K by linearity. Let K_U denote the simplex K with the metric it inherits as a subspace of l_2. We will refer to any such simplex as a uniform simplicial complex. Gromov's third definition is asdim$X \leq n$ if and only if for every $\epsilon > 0$, there is a uniformly cobounded, ϵ-Lipschitz map from X to a uniform simplicial complex of dimension n.

To any open cover $\mathcal{U}$ there is an associated simplicial complex, the *nerve*, defined by mapping each element of the cover to a vertex. An edge spans two vertices precisely when the corresponding elements of the cover intersect nontrivially. More generally, an n-simplex will span any collection of $n+1$ vertices corresponding to $n+1$ elements of the cover whose intersection is nontrivial. We will consider the nerve $N(\mathcal{U})$ in the uniform metric. Thus, there is a natural n-dimensional simplicial complex associated to a cover of multiplicity $n+1$.

The canonical projection to the nerve $p : X \to N(\mathcal{U})$ is defined by the partition of unity $\{\phi_U : X \to \mathbb{R}\}_{U \in \mathcal{U}}$, where $\phi_U(x) = d(x, X\backslash U)/\sum_{V \in \mathcal{U}} d(x, X\backslash V)$. The family $\{\phi_U : X \to \mathbb{R}\}_{U \in \mathcal{U}}$ defines a map p to the Hilbert space $l_2(\mathcal{U})$ with basis indexed by $\mathcal{U}$. The nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is realized in $l_2(\mathcal{U})$ by taking every vertex U to the corresponding element of the basis. Clearly, the image of p lies in the nerve.

Example. Let X be an infinite tree. Then asdim$X = 1$. Indeed, fix some vertex $x_0 \in X$, and declare this to be the root. Now, for a given $\epsilon > 0$, take $1/N < \epsilon$. Let T be an infinite tree in l_2. Let t_0 be a vertex of T. Map each geodesic ray in X based at x_0 to a geodesic ray in T based at t_0 by sending $\{x_0, x_{\alpha_1}, \ldots\}$ to $\{t_0, t_{\alpha_1'}, \ldots\}$ by mapping segments of length N to vertices in the following way:

$$\{x_0, x_{\alpha_1}, \ldots x_{\alpha_{N-1}}\} \mapsto t_0, \{x_0, x_{\alpha_N}, \ldots x_{\alpha_{2N-1}}\} \mapsto t_{\alpha_1}, \ldots.$$

Clearly this map is $1/N$-Lipschitz, and N-uniformly cobounded. Thus, by the third definition of asdim, the tree has dimension no bigger than 1. Since the tree contains a geodesic path $\{x_0, x_1, \ldots\}$, with $d(x_i, x_{i+1}) = 1$, asdim$X > 0$.

There are several notions of equivalence in the asymptotic sense. For metric spaces X and Y we write $X \sim_{Hau} Y$ if the Hausdorff distance between X and Y is finite. Call two metrics dist$_1$ and dist$_2$ on the same space X *Lipschitz equivalent* if the maps dist$_1/dist_2$ and dist$_2$/dist$_1$ are bounded as functions $X \times X \setminus \Delta_X \to \mathbb{R}$, where Δ_X denotes the diagonal $(x, x) \in X \times X$. We write $X \sim_{Lip} Y$ if there is a bijection $X \to Y$ which brings dist$_X$ to a metric on Y which is Lipschitz equivalent to the original metric dist$_Y$. Call the metric spaces X and Y quasi-isometric if there exist metric spaces X' and Y' such that $X \sim_{Hau} X' \sim_{Lip} Y' \sim_{Hau} Y$. Asymptotic dimension is an invariant of quasi-isometry.

Asymptotic dimension is also an invariant of the coarse category introduced by Roe [15]. Objects in the coarse category are proper metric spaces, i.e., metric spaces in which every closed r-ball, $B_r(x)$ is compact. A morphism is a (not necessarily continuous) map $f : X \to Y$ for which every bounded set in Y has bounded inverse image in X and such that for every $R > 0$ there is an $S > 0$ such that

$$d(x, x') \leq R \Rightarrow d(f(x), f(x')) \leq S.$$

Two maps f and $g : X \to Y$ are *coarsely equivalent* if there is a $K > 0$ such that $d(f(x), g(x)) \leq K$ for all $x \in X$. Two spaces X and Y are said to be coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are coarsely equivalent to the identity maps on X and Y, respectively.

2.2 The Word Metric on a Finitely Generated Group

As mentioned in the introduction, there is a left invariant metric called the word metric which has the property that $d(s, e) = 1$ for all generators s, where e is the group identity. We describe this metric in more detail presently.

Let Γ be a finitely generated group. Fix a finite set of generators S and require that $S = S^{-1}$. Define a norm on Γ with respect to the fixed generating set S by setting $\|a\|_S$ equal to the minimal length of any S-word which presents the element $a \in \Gamma$ with the convention that $\|e\| = 0$. It is easy to check that $\|ab\| \leq \|a\| + \|b\|$. Now, define the word metric associated to the generating set S by $\mathrm{dist}_S(x,y) = \|x^{-1}y\|_S$. Notice that we take $d(x,y) = \|x^{-1}y\|$ so that multiplication from the left is an isometry. Indeed, $d(\alpha x, \alpha y) = \|x^{-1}\alpha^{-1}\alpha y\| = \|x^{-1}y\| = d(x,y)$.

If S' is any other generating set for Γ, such that $(S')^{-1} = S'$, then dist_S and $\mathrm{dist}_{S'}$ are Lipschitz equivalent, with Lipschitz constant equal to the longest word in the dictionary of translation between S and S'. Thus, $\mathrm{asdim}(\Gamma, \mathrm{dist}_S) = \mathrm{asdim}(\Gamma, \mathrm{dist}_{S'})$, so $\mathrm{asdim}\Gamma$ is well-defined without reference to a generating set. Thus, we speak of the asdim of a finitely generated group.

The local structure of the metric space $|\Gamma|_S$ depends on the generating set, as it is simply the Cayley graph associated to the generating set. However, the asymptotic approach yields a geometric object associated to the group invariant of the choice of generators. This is an advantage of this approach.

2.3 Finite Asymptotic Dimension

As mentioned in the introduction, results on the finiteness of asdim of finitely generated groups became of interest following a paper of Yu [18] in which he proves the coarse Baum-Connes Conjecture holds, and thus (by the so-called Descent Principle) the Novikov Higher Signature Conjecture holds for certain manifolds whose fundamental groups have finite asdim.

We mentioned in the introduction that Gromov [11] showed that hyperbolic groups have finite asdim. A metric space X is called *δ-hyperbolic* if the δ-neighborhood of any two edges of a geodesic triangle in X contains the third. The space X is called *hyperbolic* if it is δ-hyperbolic for some $\delta > 0$. A group Γ is *hyperbolic* if its Cayley graph $C_S(\Gamma)$ with respect to some choice of generators is hyperbolic.

Next, recall that a *Coxeter group* is a finitely generated group W with the only relations of the type $(m_i m_j)^{r_{ij}} = 1$, where $r_{ii} = 1$, and $r_{ij} \in \{1, 2, \ldots\} \cup \{\infty\}$. The value of $r_{ij} = \infty$ if it is the case that no relation exists. Dranishnikov and Januszkiewicz [8] proved that Coxeter groups have finite asymptotic dimension.

It was unknown for some time whether there exist finitely generated groups with infinite asdim. If a group has finite asdim then it embeds uniformly in Hilbert space. Gromov and others have shown recently that it is possible to embed graphs called expanders in the Cayley graph of a finitely generated group. It is known that these expanders do not embed uniformly in Hilbert space, so the Cayley graph containing them will also not embed. As the asdim of a group and its Cayley graph are the same, this provides an example of a finitely generated group which does not have finite asdim (moreover, this group does not have property A; see section 2.4). Therefore determining whether certain finitely generated groups have finite asdim became important.

2.4 Property A

Guoliang Yu introduced property A for discrete groups by means of a Følner-type condition [19]. He showed that a group Γ with property A admits a uniform embedding into Hilbert space, and that this implies the coarse Baum-Connes Conjecture and the Novikov Conjecture for Γ. Recall that a function $f : X \to Y$ between metric spaces is a *uniform embedding* if for every $R > 0$ there is an $S > 0$ so that $d(z, w) \leq R$ implies that $d(f(z), f(w)) \leq S$ and $d(f(z), f(w)) \leq R$ implies $d(z, w) \leq S$. Thus, in the language of coarse geometry, f is a coarse equivalence from X to $f(X)$.

Yu's original definition is the following. A (countable) discrete metric space X is said to have property A if for every $x \in X$, there exists a sequence A_x^n of finite subsets of $X \times \mathbb{N}$ such that

1. for every n there is an $R = R(n)$ so that $A_x^n \subset B_R(x) \times \mathbb{N}$, and

2. for every $K > 0$,
$$\lim_{n\to\infty} \sup_{d(z,w)<K} \frac{|A_z^n \triangle A_w^n|}{|A_z^n \cap A_w^n|} = 0.$$

Since it appeared, there have been many different reformulations of property A. We found the following definition, due to Higson-Roe [13], easy to work with.

Let X be a discrete metric space. Let $\mathrm{Prob}(X)$ denote the space of probability measures on X with the l_1 metric. A space X is said to have property A if there exists a family of functions
$$a^n : X \to \mathrm{Prob}(X)$$

satisfying:

1. for every $n > 0$ there is an $R = R(n) > 0$ such that
$$\mathrm{supp}(a_x^n) \subset B_R(x)$$
for all $x \in X$; and

2. for every $K > 0$,
$$\lim_{n\to\infty} \sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 = 0.$$

If (X, d) and (Y, d') are coarsely equivalent metric spaces then (X, d) has property A if and only if (Y, d') has property A [17]. Thus, we may ask whether a group has property A without reference to a generating set.

Example. Let X be an infinite tree. Then, X has property A. Indeed, using Yu's definition, fix a geodesic ray γ_0 based at some fixed $x_0 \in X$. For every other x let γ_x denote that geodesic ray such that $\gamma_x \cap \gamma_0$ is also a geodesic ray. Then, put A_x^n equal to $\{\text{the first } n \text{ vertices of } \gamma_x\} \times 1$. Clearly, A_x^n is a finite subset of $X \times \mathbb{N}$. Conditions 1 and 2 are trivial to verify.

For a subset F of a group Γ, denote by $\partial_d F$ those elements of $\Gamma \setminus F$ which are within distance d of F. Then, Følner's condition for amenability can be phrased as follows. The group Γ is *amenable* if and only if it there exists an exhaustion of Γ by finite sets $F_1 \subset F_2 \subset \cdots \Gamma$, such that

$$\limsup_{n\to\infty} \frac{|\partial_d F_n|}{|F_n|} = 0,$$

for every $d > 0$. It is known that amenable groups, and hyperbolic groups have property A. Actually it is known that both of these classes of groups have finite asdim. A discrete metric space X has *bounded geometry* if for every $r > 0$ there is a number $c(r) > 0$ such that $\mathrm{Card}(B_r(x)) \leq c(r)$, for every $x \in X$. Finitely generated groups in the word metric have bounded geometry. Higson-Roe [13] showed that metric spaces with bounded geometry (see section 3.1) and finite asdim have property A. In particular, then, finitely generated groups with finite asdim have property A. It is also easy to see that the direct product of two groups with property A has property A. It is known that the semi-direct product of two groups with property A will also have property A (see [17]).

2.5 Free Products with Amalgamation and HNN Extensions

Free products and amalgamated free products arise in basic algebraic topology in connection with the Seifert-van Kampen theorem. For the reader's convenience, we recall the theorem here.

Theorem 2.5.1 (Seifert van-Kampen) *Let $X = U \cup V$ with U, V and $U \cap V$ all nonempty, open and arcwise connected. Then, the canonical maps of the fundamental groups of U, V and $U \cap V$ into X induce an isomorphism*

$$\Theta : \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X).$$

The free product has a very simple description in terms of generators and relations. Let $A = \langle S_A \mid R_A \rangle$ and $B = \langle S_B \mid R_B \rangle$ be presentations for the groups A and B. Then, the free product $A * B$ has a presentation $A * B = \langle S_A \amalg S_B \mid R_A \amalg R_B \rangle$.

In Algebraic Topology, this corresponds to computing the fundamental group of a one-point union.

If A and B are groups and C is a group equipped with two injective homomorphisms $\phi_A : C \to A$ and $\phi_B : C \to B$, then we can form the amalgamated free product of A and B over C, denoted $A *_C B$, by taking $A *_C B = (A * B)/N$, where N is the normal subgroup of $A * B$ generated by all elements of the form $\phi_A(c)\phi_B(c)^{-1}$, where $c \in C$. This corresponds to computing fundamental groups of two complexes glued together over a common part, as in the theorem above.

The HNN (for G. Higmann, B.H. Neumann and H. Neumann) extension also has a simple combinatorial description. Let G be a group with $A \subset G$ a subgroup. Suppose that $\phi : A \to G$ is an injective homomorphism. Then the HNN extension $G*_A$ has the following presentation $\langle G, t \mid t\phi(a)t^{-1} = a\rangle$. The HNN extension corresponds to the fundamental group of the mapping torus. It also has a place in history as it was used by P.S. Novikov to prove the word problem was unsolvable.

2.6 Graphs of Groups

We recall some of the basic constructions and known facts about graphs of groups. Our development and notation closely follows that of Serre [16]. Let Y be a non-empty, connected oriented graph. Associate a group G_P to each $P \in \mathrm{Vert}Y$. For an edge y, denote by $\bar{y}$ the edge with opposite orientation. Then, associate a group $G_y = G_{\bar{y}}$ equipped with two injective homomorphisms, $\phi_y : G_y \to G_{t(y)}$ and $\phi_{\bar{y}} : G_{\bar{y}} \to G_{i(y)}$, to each $y \in \mathrm{Edge}Y$.

Define an auxiliary group $F(G, Y)$ to be the group generated by the elements of the G_P and the elements $y \in \mathrm{Edge}Y$ subject to the relations:

$$\bar{y} = y^{-1}$$

and

$$y\phi_y(a)y^{-1} = \phi_{\bar{y}}(a),$$

if $y \in \mathrm{Edge}Y$ and $a \in G_y$. More formally, one can take $F(G,Y)$ to be the free product of the vertex groups G_P and the free group generated by the edges, with the two relations described above. Let c be a reduced path in Y starting at some vertex P_0. Let $y_1, y_2, \ldots, y_n$ denote the edges associated to c, where $t(y_i) = P_i$. Then the length of c is n, and we write $i(c) = G_{P_0}$ and $t(c) = G_{P_n}$. A word of type c in $F(G,Y)$ is a pair (c,μ) where c is a path as above and μ is a sequence $r_0, r_1, \ldots, r_n$, where $r_i \in G_{P_i}$, The associated element of the group $F(G,Y)$ is

$$|c,\mu| = r_0 y_1 r_1 y_2 r_2 \cdots y_n r_n \in F(G,Y).$$

Serre gives two equivalent definitions of the fundamental group of the graph of groups (G,Y). For the first description, let P_0 denote a fixed vertex and G_{P_0} the associated group. Let $\pi = \pi_1(G,Y,P_0)$ be the set of elements of $F(G,Y)$ associated to a path c in Y with $i(c) = t(c) = G_{P_0}$. Obviously, $\pi \subset F(G,Y)$ is a subgroup. For the second description, let T be a maximal subtree of Y, and define $\pi = \pi_1(G,Y,T)$ to be the quotient of $F(G,Y)$ by the normal subgroup generated by the elements $t \in \mathrm{Edge}T$. If g_y denotes the image of $y \in \mathrm{Edge}Y$ in π, then the group π is the group generated by the groups G_P and the elements g_y subject to the relations

$$g_{\bar{y}} = g_y^{-1},$$

$$g_y \phi_y(a) g_y^{-1} = \phi_{\bar{y}}(a),$$

and

$$g_t = 1$$

where $a \in G_y$, and $t \in \mathrm{Edge}T$. So, in particular, $\phi_t(a) = \phi_{\bar{t}}(a)$ for all $t \in \mathrm{Edge}T$. The equivalence of the descriptions is proven in Serre [16].

Examples.

(1) If (G,Y) is a graph of groups where every vertex group is the trivial group, then $\pi_1(F,G,T)$ is simply the fundamental group of the graph Y.

(2) If Y is the graph with two vertices P, Q and one edge y, then

$$\pi_1(G, Y, P) = \pi_1(G, Y, Q) = G_P *_{G_y} G_Q,$$

the free product of G_P and G_Q amalgamated over G_y.

(3) If Y is the graph with one vertex P and one edge y then $\phi_{\bar{y}}(G_y)$ is a subgroup of G_P, and $\pi_1(G, Y, P)$ is precisely the HNN extension of G_P over the subgroup $\phi_{\bar{y}}(G_y)$ by means of $\phi_y \phi_{\bar{y}}^{-1}$.

We now describe the construction of the so-called Bass-Serre tree $\tilde{X}$ on which π will act by isometries. Let T be a maximal tree and let π_P denote the canonical image of G_P in π, obtained via conjugation by the path c, where c is the unique path in T from the basepoint P_0 to the vertex P. Then, set

$$\text{Vert}\tilde{X} = \coprod_{P \in \text{Vert}Y} \pi / \pi_P.$$

From now on we will implicitly identify the edge groups with their images in the group $F(G, Y)$.

An edge in $\tilde{X}$ connects the vertex $xG_{i(y)}$ to the vertex $xyG_{t(y)}$ for all $y \in \text{Edge}Y$ and all $x \in \pi$. Observe that the stabilizers of the vertices are conjugates of the corresponding vertex groups, while the stabilizer of the edge is $xy\phi_y(G_y)y^{-1}x^{-1}$, a conjugate of the image of the vertex group. This obviously stabilizes the second vertex, and it stabilizes the first vertex since $y\phi_y(G_y)y^{-1} = \phi_{\bar{y}}(G_y) \subset G_{i(y)}$. It is known that the action of left multiplication on $\tilde{X}$ is isometric.

In order to consider the fundamental group of a graph of groups as a metric space, we assume that the graph Y is finite and that the groups associated to the edges and vertices are finitely generated with some fixed set of generators chosen for each group. We let S denote the disjoint union of the generating sets for the groups, and require that $S = S^{-1}$. By the norm $\|x\|$ of an element $x \in G_P$ we mean the minimal number of generators in the fixed generating set required to present the element x. We endow each of the groups G_P with the word metric given by $d(x, y) = \|x^{-1}y\|$.

We extend this metric to the group $F(G, Y)$ and hence to the subgroup $\pi_1(G, Y, P_0)$ in the natural way, by adjoining to S the collection $\{y, y^{-1} \mid y \in \text{Edge}Y\}$.

The structure theorem of Bass-Serre says that groups acting co-compactly on trees by isometries correspond bijectively to fundamental groups of finite graphs of groups. Thus, we will view the fundamental group of a graph of groups over a finite, connected graph as a group acting on a tree by isometries with compact quotient.

Definition. Let Γ act on the metric space X by isometries. For every $R > 0$ denote by $W_R(x)$ the R-stabilizer of the point $x \in X$. Specifically, $W_R(x) = \{\gamma \in \Gamma \mid d_X(\gamma x, x) \leq R\}$.

Proposition 2.6.1 *Let Y be a non-empty, finite, connected graph, and (G, Y) the associated graph of finitely generated groups. Let P_0 be a fixed vertex of Y; then under the action of π on $\tilde{X}$, the R-stabilizer $W_R(P_0)$ is precisely the set of elements of type c in $F(G, Y)$ with $l(c) \leq R$.*

Proof: Let $K \subset F(G, Y)$ be the collection of those words of type c in $F(G, Y)$ whose associated path begins at P_0. Then every $x = |c, \mu| \in K$ with $l(c) \leq R$ is in $W_R(G_{P_0})$ since $\text{dist}_{\tilde{X}}(xG_{P_0}, G_{P_0})$ is the number of edges in the path c, which is at most R. For the reverse inclusion, it suffices to show that given a reduced word $x = |c, \mu|$, with $l(c) = R$, the length of any coset representative xG_{P_0} is also R. Consider the word $xg = r_0 y_1 r_1 \cdots y_n r_n g$, where $y_i \in \text{Edge}Y$, $r_i \in G_{P_i}$ and $g \in G_{P_0}$. The relations on $F(G, Y)$ allow for only one type of reduction involving edges to occur. It occurs in one of the following two forms:

1. $y\phi_y(a) = \phi_{\bar{y}}(a)y$, or
2. $y\phi_y(a)y^{-1} = \phi_{\bar{y}}(a)$,

where $a \in G_y$ and $y \in \text{Edge}Y$. Clearly only a reduction of type (2) can decrease the (path) length of the word. So, assume that there are edges $y_{j+1}^{-1} = y_j$ and through a sequence of type-(1) reductions, x is translated to

$$r_0 y_1 \cdots y_j r_j \phi_{y_j}(b) y_{j+1} r'_{j+1} \cdots y_n r'_n.$$

Now, in order for a type-(2) reduction to occur here, we must have $r_j \phi_{y_j}(b) = \phi_{y_j}(d)$ for some $d \in G_{y_j}$. This implies that $r_j = \phi_{y_j}(db^{-1})$, which enables a type-(2) reduction of the original word x. As x was assumed to be reduced, this cannot occur. ∎

2.7 Complexes of Groups

Complexes of groups were introduced by A. Haefliger as an attempt to describe actions of groups on simply connected simplicial complexes in the same way that graphs of groups describe the action of groups on trees. A problem that arises is that the quotient of a simplicial complex by a simplicial action may identify faces of simplices. Thus, it may not be the case that the quotient is a simplicial complex. We avoid this problem by introducing combinatorial substitutes for simplicial complexes called small categories without loops. Our notation and development will follow Bridson-Haefliger [5].

Definition. A small category without loops (briefly a SCWOL) is a set $\mathcal{X}$ which is the disjoint union of two sets:

$$\mathcal{X} = V(\mathcal{X}) \coprod E(\mathcal{X}).$$

The elements of $V(\mathcal{X})$ are called the vertices and the elements of $E(\mathcal{X})$ are called the edges. There are maps

$$i : E(\mathcal{X}) \to V(\mathcal{X})$$

and

$$t : E(\mathcal{X}) \to V(\mathcal{X})$$

which assign to each edge a the initial vertex of a and the terminal vertex of a, respectively. Let $E^{(2)}(\mathcal{X}) = \{(a, b) \in E(\mathcal{X}) \times E(\mathcal{X}) \mid i(a) = t(b)\}$ denote the pairs of composable edges. There is also a map

$$E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$$

which assigns to each pair (a, b) an edge (ab) called the composition of a and b. These maps are required to satisfy

1. $i(ab) = i(b)$, and $t(ab) = t(a)$ for all $(a, b) \in E^{(2)}(\mathcal{X})$;

2. $a(bc) = (ab)c$ for all edges a, b and c with $i(a) = t(b)$ and $i(b) = t(c)$; and

3. $i(a) \neq t(a)$.

Let $E^{(k)}(\mathcal{X})$ denote the composable sequences of edges of length k; that is, $E^{(k)}(\mathcal{X}) = \{(a_1, \ldots, a_k) \in E^k \mid i(a_i) = t(a_{i+1}), \text{ for } i = 1, \ldots, k-1\}$. By convention, $E^{(0)}(\mathcal{X}) = V(\mathcal{X})$. We define the *dimension* of the SCWOL $\mathcal{X}$ to be the maximum k such that $E^{(k)}(\mathcal{X})$ is not empty.

Definition. The geometric realization $|\mathcal{X}|$ is a piecewise Euclidean polyhedral complex, with each k-cell isometric to the standard simplex Δ^k. There is one such k-simplex A for each $A \in E^{(k)}(\mathcal{X})$. The identifications are the obvious ones, induced by the face relation among simplices.

Observe that the geometric realization need not be a simplicial complex, since it may be the case that the intersection of two simplices is a union of faces. One can eliminate this problem by taking the barycentric subdivision, if one requires simplicial complexes. The geometric dimension of the geometric realization of the SCWOL $\mathcal{X}$ is the same as the combinatorial dimension of the combinatorial object $\mathcal{X}$.

Definition. A group action on a SCWOL is a homomorphism $G \to Aut(\mathcal{X})$ satisfying

1. for every $g \in G$, and for all $a \in E(\mathcal{X})$ $g.i(a) \neq t(a)$.

2. for every $g \in G$, and for all $a \in E(\mathcal{X})$ if $g.i(a) = i(a)$, then $g.a = a$.

Notice that a group action on a SCWOL induces an isometric action of the group on the geometric realization $|\mathcal{X}|$. Since we are primarily concerned with isometric actions on metric spaces, this is the action that we consider.

One forms the quotient $\mathcal{Y} = G\backslash\mathcal{X}$ of the SCWOL $\mathcal{X}$ by the action of G by taking $V(\mathcal{Y}) = G\backslash V(\mathcal{X})$, and $E(\mathcal{Y}) = G\backslash E(\mathcal{X})$. One can verify that $\mathcal{Y}$ has the structure of a SCWOL.

Definition. A complex of groups $G(\mathcal{Y})$ over a SCWOL $\mathcal{Y}$ is a collection $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ satisfying

1. to each $\sigma \in V(\mathcal{Y})$, there corresponds a group G_σ called the local group at σ;
2. to each $a \in E(\mathcal{Y})$ there corresponds an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$; and
3. For each $(a, b) \in E^{(2)}(\mathcal{Y})$, there is a $g_{a,b} \in G_{t(a)}$ such that
   (i) $Ad(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $Ad(g_{a,b})$ denotes conjugation by $g_{a,b}$, and
   (ii) $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$, for all $(a, b, c) \in E^{(3)}(\mathcal{Y})$.

Given a group G and an action of G on the SCWOL $\mathcal{X}$, there is an explicit construction of the complex of groups over the quotient SCWOL which we do not describe here. On the other hand, it is not always the case that an arbitrary complex of groups can be associated to a group action on some SCWOL $\mathcal{X}$. When this occurs, we say that the complex of groups is *developable* and we refer to the associated SCWOL $\mathcal{X}$ as the development of $G(\mathcal{Y})$.

It is clear that SCWOLs of dimension 1 must have precisely two types of vertices: sources and sinks. A source is an initial vertex of every edge it is contained in and a sink is a terminal vertex of every edge it is contained in. Every one-dimensional simplicial complex (graph) can be given the structure of a one-dimensional SCWOL by placing a source vertex in the middle of every edge, thus giving the original vertices the structure of sinks. It is easy to verify that the theory of complexes of groups over one-dimensional SCWOLs is precisely the same as the theory of graphs of groups. Phrased in terms of the language of complexes of groups, the Bass-Serre structure theorem for groups acting without inversion on graphs says that if $dim(\mathcal{Y}) = 1$, then $G(\mathcal{Y})$ is always developable.

When a complex of groups is developable, there is an explicit method of constructing both the SCWOL $\mathcal{X}$ and the group G which acts on the SCWOL. The SCWOL $\mathcal{X}$ on which the group acts is simply connected and has an explicit description in a similar way to the construction of the tree $\tilde{X}$ in the theory of graphs of groups.

Indeed, if $G(\mathcal{Y})$ is a developable complex of groups, then we can define the development $D(\mathcal{Y})$ to be the SCWOL whose vertices and edges are given by $V(D(\mathcal{Y})) = \{gG_\sigma\}$, and $E(D(\mathcal{Y})) = \{gG_{i(a)}\}$. Then the group G acts on the development $D(\mathcal{Y})$ by left multiplication. The development is isomorphic to the SCWOL $\mathcal{X}$, given above.

We describe the fundamental group of the complex of groups $\pi_1(G(\mathcal{Y}))$ which is the group G, up to isomorphism. As in the theory of graphs of groups, there are two equivalent descriptions of the fundamental group. Both rely on the construction of the auxiliary group $FG(\mathcal{Y})$. Let $E^{\pm}(\mathcal{Y})$ denote the collection of symbols $\{a^+, a^-\}$ where $a \in E(\mathcal{Y})$. Then, define $FG(\mathcal{Y})$ to be the free product of the local groups G_σ and the free group generated by the collection $E^{\pm}(\mathcal{Y})$ subject to the additional relations:

1. $(a^+)^{-1} = a^-$, and $(a^-)^{-1} = a^+$;
2. $a^+b^+ = g_{a,b}(ab)^+$;
3. $\psi_a(g) = a^+ga^-$, for all $g \in G_{i(a)}$.

The first description of the fundamental group is in terms of $G(\mathcal{Y})$-loops based at some fixed vertex σ_0. In $E^{\pm}(\mathcal{Y})$, define $i(a^+) = i(a)$, $i(a^-) = t(a)$, $t(a^+) = t(a)$, and $t(a^-) = i(a)$. Then, an edge path in $\mathcal{Y}$ is a sequence $(e_1, \ldots, e_k)$ with $t(e_i) = i(e_{i+1})$, for all $i = 1, \ldots, k-1$. A $G(\mathcal{Y})$-path issuing from σ_0 is a sequence $(g_0, e_1, g_1, \ldots, e_k, g_k)$, where $(e_1, \ldots, e_k)$ is an edge path in $\mathcal{Y}$, and $g_0 \in G_{\sigma_0}$, and $g_i \in G_{t(e_i)}$, for $i > 0$. We associate the word $g_0e_1 \ldots e_kg_k \in FG(\mathcal{Y})$ to the path described above. A $G(\mathcal{Y})$-loop based at σ_0 is a $G(\mathcal{Y})$ path with $t(e_k) = \sigma_0$. There is an obvious notion of homotopy on $G(\mathcal{Y})$-paths, i.e., the notion of homotopy on the geometric realization. The fundamental group $\pi_1(G(\mathcal{Y}), \sigma_0)$ is the collection of all words associated to $G(\mathcal{Y})$-loops based at σ_0, up to homotopy equivalence.

The second description is much simpler. Let T be a maximal tree in $|\mathcal{Y}|^{(1)}$. Then, $\pi_1(G(\mathcal{Y}), T)$ is $FG(\mathcal{Y})$ subject to the additional relation $a^+ = 1$, for all $a \in T$. For a connected SCWOL, there is an isomorphism $\pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G(\mathcal{Y}), T)$.

The following proposition generalizes Proposition 2.6.1.

Proposition 2.7.1 *Let $G(\mathcal{Y})$ be a developable complex of groups. Fix a vertex $\sigma_0 \in \mathcal{Y}$, and consider the action of $\pi_1(G(\mathcal{Y}), \sigma_0)$ on the simply connected SCWOL $\mathcal{X}$ induced by the complex of groups. Then the R-stabilizer $W_R(\sigma_0)$ is the set of all elements with associated path c of length not exceeding R.*

Proof: Let $K \subset FG(\mathcal{Y})$ denote the set of all reduced words in $FG(\mathcal{Y})$ with associated path c issuing from σ_0. Let $K_R \subset K$ denote those reduced words whose path c has length equal to R. Since the path c lifts to a path in the development $\mathcal{X}$, $dist_{\mathcal{X}}(xG_{\sigma_0}, G_{\sigma_0})$ is at most the number of edges in the path R.

Thus, to prove that $W_R(\sigma_0) = K_R$, it suffices to show that the length of any coset representative of xG_{σ_0} with $x \in K_R$ is also R. To this end, we consider the word $xg = g_0e_1 \cdots e_Rg_Rg$, where $g \in G_{\sigma_0}$. We are assuming the x-path is reduced, and we must show that multiplication by g does not shorten this path.

We enumerate the relations on $FG(\mathcal{Y})$ as follows:

1. the relations on the G_σ;
2. $(a^+)^{-1} = a^-$, and $(a^-)^{-1} = a^+$;

3. $(a^+b^+) = g_{a,b}(ab)^+$; and

4. $\psi_a(g) = a^+ga^-$, for $g \in G_{i(a)}$.

Only the relations (3) and (4) can change the length of the path. We have seen that relations (1),(2) and (4) alone cannot decrease the path length (see Proposition 2.6.1), so it suffices to show that a reduction involving relation (3) cannot occur. To this end, suppose that a sequence of relations occur which transform xg into $g_0e_1 \cdots g_{k-1}e_kg_kg'e_{k+1}g'_k \cdots g'_R$. In order for a type-(3) reduction to occur, the edges e_k and e_{k+1} must be composable. But, in this case, we can obtain a reduction of the original path x through a type-(4) and then type-(3) reduction. Since we assumed that this path was already reduced, no type-(3) reduction can occur. Thus, $K_R = W_R(\sigma_0)$. ∎

CHAPTER 3
FINITE DIMENSIONALITY RESULTS

3.1 Union Theorems

In this section we formulate two union theorems for asdim which we found useful.

Definition. A family of metric spaces $\{F_\alpha\}$ satisfies the inequality asdim$F_\alpha \leq n$ *uniformly* if for arbitrarily large $d > 0$ there exist R-bounded d-disjoint families $\mathcal{U}_\alpha^0 \ldots \mathcal{U}_\alpha^n$ of subsets of F_α such that the union $\cup_i \mathcal{U}_\alpha^i$ is a cover of F_α for some $R > 0$.

A typical example of such a family is the case where all F_α are isometric to a space F with asdim$F \leq n$.

Theorem 3.1.1 *Assume that* $X = \cup_\alpha F_\alpha$ *and* asdim$F_\alpha \leq n$ *uniformly. Suppose that for any* r *there exists* $Y_r \subset X$ *with* asdim$Y_r \leq n$ *and such that the family* $\{F_\alpha \setminus Y_r\}$ *is* r*-disjoint. Then* asdim$X \leq n$.

As an immediate corollary, we obtain a finite union theorem.

Theorem 3.1.2 (Finite Union Theorem) *Suppose that a metric space is presented as a union* $A \cup B$ *of subspaces. Then* asdim$A \cup B \leq \max\{\text{asdim}A, \text{asdim}B\}$.

Proof: We apply Theorem 3.1.1 to the case when the family of subsets consists of A and B and we take $Y_r = B$. ■

Recall that in the theory of covering dimension, an n-dimensional space can be decomposed into a union of $n+1$ 0-dimensional sets. For a more detailed comparison of asymptotic and classical phenomena, see Dranishnikov [6].

The proof of Theorem 3.1.1 is based on the idea of saturation of one family by the other. Let $\mathcal{V}$ and $\mathcal{U}$ be two families of subsets of a metric space X.

Definition. For $V \in \mathcal{V}$ and $d > 0$ we denote by $N_d(V;\mathcal{U})$ the union of V and all elements $U \in \mathcal{U}$ with $d(V,U) = \min\{d(x,y) \mid x \in V, y \in U\} \leq d$. By *d-saturated union* of $\mathcal{V}$ and $\mathcal{U}$ we mean the following family $\mathcal{V} \cup_d \mathcal{U} = \{N_d(V;\mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(U,V) > d \ \ for \ \ all \ \ V \in \mathcal{V}\}$.

Note that this is not a commutative operation. Also note that $\{\emptyset\} \cup_d \mathcal{U} = \mathcal{U}$ and $\mathcal{V} \cup_d \{\emptyset\} = \mathcal{V}$ for all d.

Proposition 3.1.3 *Assume that $\mathcal{U}$ is d-disjoint and R-bounded, $R \geq d$. Assume that $\mathcal{V}$ is $5R$-disjoint and D-bounded. Then $\mathcal{V} \cup_d \mathcal{U}$ is d-disjoint and $D+2(d+R)$-bounded.*

Proof: First we note that elements of type U are d-disjoint in the saturated union. The same is true for elements of type U and $N_d(V;\mathcal{U})$. Now consider elements $N_d(V;\mathcal{U})$ and $N_d(V';\mathcal{U})$. Note that they are contained in the $d+R$-neighborhoods of V and V' respectively. Since V and V' are $5R$-disjoint, and $R \geq d$, the d-neighborhoods will be d-disjoint.

Clearly, $diam N_d(V;\mathcal{U}) \leq diam V + 2(d+R) \leq D + 2(d+R)$. ■

Proof: (Proof of Theorem 3.1.1). Let d be given. Consider the number R and the families $\mathcal{U}_\alpha^0 \dots \mathcal{U}_\alpha^n$ from the definition of the uniform inequality $\text{asdim} F_\alpha \leq n$. We may assume that $R > d$. We take $r = 5R$ and take Y_r satisfying the hypotheses of the Theorem. Consider r-disjoint, D-bounded families $\mathcal{V}^0, \dots, \mathcal{V}^k$ from the definition of $\text{asdim} Y_r \leq k$. Let $\bar{\mathcal{U}}_\alpha^i$ be the restriction of $\mathcal{U}_\alpha^i$ over $F_\alpha \setminus Y_r$, i.e. $\bar{\mathcal{U}}_\alpha^i = \{U \setminus Y_r \mid U \in \mathcal{U}_\alpha^i\}$. Let $\bar{\mathcal{U}}^i = \cup_\alpha \bar{\mathcal{U}}_\alpha^i$. Note that the family $\bar{\mathcal{U}}^i$ is d-disjoint and R-bounded. For every i we define $\mathcal{W}^i = \mathcal{V}^i \cup_d \bar{\mathcal{U}}^i$. By Proposition 3.1.3 the family $\mathcal{W}^i$ is d-disjoint and uniformly bounded. Clearly $\cup_i \mathcal{W}^i$ covers X. ■

3.2 Groups Acting on Finite Dimensional Spaces

Assume that a group Γ acts on a metric space X. Recall that the R-stabilizer $W_R(x_0)$ of a point $x_0 \in X$ is the set of all $g \in \Gamma$ with $g(x_0) \in B_R(x_0)$. Here $B_R(x)$ denotes the closed ball of radius R centered at x.

Theorem 3.2.1 *Assume that a finitely generated group* Γ *acts by isometries on a metric space* X *with a base point* x_0 *and with* $\text{asdim} X \leq k$*. Suppose also that* $\text{asdim} W_R(x_0) \leq n$ *for all* R*. Then* $\text{asdim}\Gamma \leq (n+1)(k+1) - 1$.

Proof: We define a map $\pi : \Gamma \to X$ by the formula $\pi(g) = g(x_0)$. Then $W_R(x_0) = \pi^{-1}(B_r(x_0))$. Fix a generating set $S = S^{-1}$. Let $\lambda = \max\{d_X(s(x_0), x_0) \mid s \in S\}$. We now show that π is λ-Lipschitz. Since the metric d_S on Γ is induced from the geodesic metric on the Cayley graph, it suffices to check that $d_X(\pi(g), \pi(g')) \leq \lambda$ for all $g, g' \in \Gamma$ with $d_S(g, g') = 1$. Thus, $g^{-1}g' = s$ for some $s \in S$. Hence, we may assume without loss of generality that $g' = gs$. Then $d_X(\pi(g), \pi(g')) = d_X(g(x_0), gs(x_0)) = d_X(x_0, s(x_0)) \leq \lambda$.

Note that $\gamma B_R(x) = B_R(\gamma(x))$ and $\gamma(\pi^{-1}(B_R(x))) = \pi^{-1}(B_R(\gamma(x)))$ for all $\gamma \in \Gamma$, $x \in X$ and all R.

Given $r > 0$, there are λr-disjoint, R-bounded families $\mathcal{F}^0, \ldots, \mathcal{F}^k$ on the orbit Γx_0. Let $\mathcal{V}^0, \ldots, \mathcal{V}^n$ on $W_{2R}(x_0)$ be r-disjoint uniformly bounded families given by the definition of the inequality $\text{asdim} W_R(x_0) \leq n$. For every element $F \in \mathcal{F}^i$ we choose an element $g_F \in \Gamma$ such that $g_F(x_0) \in F$. We define $(k+1)(n+1)$ families of subsets of Γ as follows

$$\mathcal{W}^{ij} = \{g_F(C) \cap \pi^{-1}(F) \mid F \in \mathcal{F}^i, C \in \mathcal{V}^j\}.$$

Since multiplication by g_F from the left is an isometry, every two distinct sets $g_F(C)$ and $g_F(C')$ are r-disjoint. Note that $\pi(g_F(C) \cap \pi^{-1}(F))$ and $\pi(g_{F'}(C) \cap \pi^{-1}(F'))$ are λr-disjoint for $F \neq F'$. Since π is λ-Lipschitz, the sets $g_F(C) \cap \pi^{-1}(F)$ and $g_{F'}(C') \cap \pi^{-1}(F')$ are r-disjoint. The families $\mathcal{W}^{ij}$ are uniformly bounded, since the families $\mathcal{V}^j$ are, and multiplication by g from the left is an isometry on Γ. We check that the union of the families $\mathcal{W}^{ij}$ forms a cover of Γ. Let $g \in \Gamma$ and let $\pi(g) = F$, i.e. $g(x_0) \in F$. Since $diam F \leq R$, $x_0 \in g_F^{-1}(F) \leq R$ and g_F^{-1} acts as an isometry, we have $g_F^{-1}(F) \subset B_R(x_0)$. Therefore, $g_F^{-1}g(x_0) \in B_R(x_0)$, i.e. $g_F^{-1}g \in W_R(x_0)$.

Hence $g_F^{-1}g$ lies in some set $C \in \mathcal{V}^j$ for some j. Therefore $g \in g_F(C)$. Thus, $g \in g_F(C) \cap \pi^{-1}(F)$. ■

The estimate $(n+1)(k+1)-1$ in Theorem 3.2.1 is far from sharp. In this chapter we are concerned with showing finite dimensionality only; we are not trying to give an exact estimate which is $n+k$. It would be difficult to get an exact estimate just working with covers. Even for proving the inequality

$$\mathrm{asdim}\Gamma_1 \times \Gamma_2 \leq \mathrm{asdim}\Gamma_1 + \mathrm{asdim}\Gamma_2$$

it is better to use a different approach to asdim (see Dransihnikov-Januszkiewicz [8]).

3.3 Graphs of Groups

Throughout this section, we will assume that the graph Y is finite and that the groups associated to the edges and vertices are finitely generated with some fixed set of generators chosen for each group. We let S denote the disjoint union of the generating sets for the groups, and require that $S = S^{-1}$. By the norm $\|x\|$ of an element $x \in G_P$ we mean the minimal number of generators in the fixed generating set required to present the element x. We endow each of the groups G_P with the word metric given by $\mathrm{dist}(x,y) = \|x^{-1}y\|$. We extend this metric to the group $F(G,Y)$ and hence to the subgroup $\pi_1(G,Y,P_0)$ in the natural way, by adjoining to S the collection $\{y, y^{-1} \mid y \in \mathrm{Edge}Y\}$.

Lemma 3.3.1 *If a group Γ acts on a tree with compact quotient and finitely generated stabilizers satisfying* $\mathrm{asdim}\Gamma_x \leq n$ *for all vertices x, then* $\mathrm{asdim}W_R(x_0) \leq n$ *for all R.*

Proof: We consider the subset $K \subset F(G,Y)$ of all words whose associated path c has $i(c) = G_{P_0}$. This set acts on the tree by left multiplication, and the fundamental group is a subset of K. We consider the R-stabilizer $W_R(P_0)$ of the fixed vertex as a subset of K, and show that this set has asymptotic dimension at most n

by induction. It will follow then that the stabilizers in the fundamental group will also have asymptotic dimension at most n.

We proceed by induction. The base case is clear since W_0 is precisely G_{P_0} and by assumption, $\text{asdim}W_0 \le n$. By Proposition 2.6.1, W_R is precisely the set of all words in K with associated path c for which $l(c) \le R$. Let $H_k \subset K$ be the set of elements whose paths have length precisely k. Then, in the above notation, $W_R = \cup_{k \le R} H_k$, so by the Finite Union Theorem it suffices to prove that $\text{asdim}H_k \le n$ for all k. Since W_0 and H_0 coincide, we proceed to the induction step on H_k.

Observe that $H_k \subset \cup_{y \in \text{Edge}Y} H_{k-1} y G_{t(y)}$. Since Y was assumed to be a finite graph, this is a finite union. So, by the Finite Union Theorem, it suffices to show that $\text{asdim}H_k \cap H_{k-1}yG_{t(y)} \le n$, for some fixed $y \in \text{Edge}Y$. Let $Y_r = H_{k-1}yN_r(\phi_y(G_y))$, where $N_r(\phi_y(G_y))$ is the r-neighborhood of $\phi_y(G_y)$ in $F(G, Y)$. Now, the set Y_r is coarsely equivalent to $H_{k-1}y\phi_y(G_y)$, and by the relations on $F(G, Y)$, this set is $H_{k-1}\phi_{\bar{y}}(G_y)y = H_{k-1}y$. Finally, $H_{k-1}y$ is coarsely equivalent to H_{k-1}, which, by the inductive hypothesis, has asymptotic dimension not exceeding n. Thus, we conclude that $\text{asdim}Y_r \le n$. Consider the sets $xyG_{t(y)}$, indexed by those $x \in H_{k-1}$, which do not end with an element of $\phi_{\bar{y}}(G_y)$. Notice that this collection does cover $H_k \cap H_{k-1}yG_{t(y)}$ since we may obtain $x\phi_{\bar{y}}(a)yg$ from $xy\phi_y(a)g$, which is of the required form. The map $G_{t(y)} \mapsto xyG_{t(y)}$ is an isometry in the word metric, so $\text{asdim}(xyG_{t(y)}) \le n$ uniformly.

To apply the Infinite Union Theorem (3.1.1), we need only show that the family $\{H_k \cap H_{k-1}yG_{t(y)} \setminus Y_r\}$ is r-disjoint. To this end, let $x \ne x'$ and suppose that z and z' are in $G_{t(y)} \setminus N_r(\phi_y(G_y))$. Consider $\text{dist}(xyz, x'yz') = \|z^{-1}y^{-1}x^{-1}x'yz'\|$. Write $z = \phi_y(a)s$, and $z' = \phi_y(a')s'$, where $\|s\|$ and $\|s'\| > r$. Then,

$$\|z^{-1}y^{-1}x^{-1}x'yz'\| = \|s^{-1}y^{-1}\phi_{\bar{y}}(a^{-1})x^{-1}x'\phi_{\bar{y}}(a')ys'\|.$$

Now, a reduction can only occur in the middle, and if $\phi_{\bar{y}}(a^{-1})x^{-1}x'\phi_{\bar{y}}(a')$ is not in $\phi_{\bar{y}}(G_y)$, then

$$\|z^{-1}y^{-1}x^{-1}x'yz'\| > \|s\| + \|s'\| > 2r.$$

So, suppose that there is a $b \in G_y$ so that $\phi_{\bar{y}}(b) = \phi_{\bar{y}}(a^{-1})x^{-1}x'\phi_{\bar{y}}(a')$. Then, putting $c = ab(a')^{-1}$, we see that $x' = x\phi_{\bar{y}}(c)$. By construction x' cannot end with a nontrivial element of $\phi_{\bar{y}}(G_y)$. Thus $\phi_{\bar{y}}(c) = e$, which implies $x' = x$, a contradiction.

We conclude that the families are r-disjoint, and therefore, by the Infinite Union Theorem, $\mathrm{asdim} H_k < n$, so that $\mathrm{asdim} W_R < n$ for every R. ■

We may apply Theorem 3.2.1 with the fact that $\mathrm{asdim} T = 1$ for all infinte trees to conclude the following theorem.

Theorem 3.3.2 *Let Γ be a fundamental group of a graph of groups with each vertex group finitely generated and with Y finite. Suppose that* $\mathrm{asdim}\Gamma_x \leq n$ *for all vertex groups Γ_x. Then,* $\mathrm{asdim}\Gamma \leq 2n+1$.

As a consequence of the examples given in section 2.6, we obtain the following corollary.

Corollary 3.3.3 *Let A and B be finitely generated groups with* $\mathrm{asdim} A \leq n$ *and* $\mathrm{asdim} B \leq n$. *Then* $\mathrm{asdim} A *_C \leq 2n+1$, *and* $\mathrm{asdim} A *_C B \leq 2n+1$. *In particular, the HNN extension and amalgamated free product preserve finite* asdim.

3.4 Complexes of Groups

Let $G(\mathcal{Y})$ be a developable complex of groups, with $\mathcal{Y}$ finite and $\mathrm{asdim}\mathcal{X} \leq k$. Suppose further that the local groups are finitely generated. Then, the finiteness of $\mathcal{Y}$ implies that the fundamental group π is finitely generated. We may apply Proposition 2.7.1 and the proof of Lemma 3.3.1 to obtain the following result.

Lemma 3.4.1 *Let π be the fundamental group of a complex of groups $G(\mathcal{Y})$ where $\mathcal{Y}$ is finite, the local groups are finitely generated, and the local groups satisfy* $\mathrm{asdim} G_\sigma \leq n$, *then* $\mathrm{asdim} W_d(\sigma_0) \leq n$.

Applying Theorem 3.2.1, we obtain the following theorem.

Theorem 3.4.2 *Let* Γ *be the fundamental group of a developable complex of groups* $G(\mathcal{Y})$ *corresponding to an action by isometries on the geometric realization of the SCWOL* $\mathcal{X}$. *Suppose that the local groups are finitely generated and that* $\text{asdim} G_\sigma \leq n$. *Assume additionally that* $\text{asdim}|\mathcal{X}| \leq k$. *Then* $\text{asdim}\Gamma \leq (n+1)(k+1) - 1$.

CHAPTER 4
ASDIM OF GROUPS ACTING ON TREES

4.1 Lipschitz Mappings to the Nerve

Let $\mathcal{U}$ be an open cover of a metric space X. Recall that the canonical projection to the nerve $p : X \to N(\mathcal{U})$ is defined by the partition of unity $\{\phi_U : X \to \mathbb{R}\}_{U \in \mathcal{U}}$, where $\phi_U(x) = d(x, X \setminus U)/\sum_{V \in \mathcal{U}} d(x, X \setminus V)$. The family $\{\phi_U : X \to \mathbb{R}\}_{U \in \mathcal{U}}$ defines a map p to the Hilbert space $l_2(\mathcal{U})$ with basis indexed by $\mathcal{U}$. The nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is realized in $l_2(\mathcal{U})$ by taking every vertex U to the corresponding element of the basis.

Proposition 4.1.1 *For every k and every $\epsilon > 0$ there exists a $\nu = \nu(\epsilon, k)$ such that for every cover $\mathcal{U}$ of a metric space X of order $\leq k+1$ with Lebesgue number $L(\mathcal{U}) > \nu$ the canonical projection to the nerve $p_{\mathcal{U}} : X \to N(\mathcal{U})$ is ϵ-Lipschitz.*

Proof. We take $\nu \geq \frac{(2k+3)^2}{\epsilon}$. Let $x, y \in X$ and $U \in \mathcal{U}$. The triangle inequality implies $|d(x, X \setminus U) - d(y, X \setminus U)| \leq d(x, y)$. Then

$$\begin{aligned} |\phi_U(x) - \phi_U(y)| \leq\ & \frac{d(x,y)}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)} + \\ & + d(y, X \setminus U) \left| \frac{1}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)} - \frac{1}{\sum_{V \in \mathcal{U}} d(y, X \setminus V)} \right| \\ & \leq \frac{1}{L(\mathcal{U})} d(x,y) + \frac{1}{L(\mathcal{U})} \Big(\sum_{V \in \mathcal{U}} |d(x, X \setminus V) - d(y, X \setminus V)| \Big) \\ & \leq \frac{2k+3}{\nu} d(x,y). \end{aligned}$$

Then

$$\begin{aligned} \|p(x) - p(y)\| &= \big(\sum_{U \in \mathcal{U}} (\phi_U(x) - \phi_U(y))^2\big)^{\frac{1}{2}} \\ &\leq \big((2k+2)(\frac{2k+3}{\nu} d(x,y))^2\big)^{\frac{1}{2}} \leq \frac{(2k+3)^2}{\nu} d(x,y) \leq \epsilon d(x,y). \blacksquare \end{aligned}$$

4.2 Simplicial Mapping Cylinders

Let X and Y be simplicial complexes. The mapping cylinder M_g of any simplicial map $g : X \to Y$ has a natural structure as a cellular complex. Closed m-cells are the m-simplices of X and Y, and the images under the quotient map q of $\sigma^{m-1} \times I$, where σ^{m-1} is an $m-1$-simplex in X. These maps $q|_{\sigma^{m-1}\times I}$ are the characteristic maps of corresponding m-cells.

An orientation on an n-simplex is an enumeration of its vertices by the numbers $0, 1, \ldots, n$. An *orientation* on an n-dimensional simplicial complex K is a map $o : K^{(0)} \to \{0, 1, \ldots, n\}$ whose restriction to every n-simplex defines an orientation on it. This map defines a simplicial map $\xi : K \to \Delta^n$ to the n-simplex which is nondegenerate on every n-simplex. The barycentric subdivision of any simplicial complex has a natural orientation. We will always assume that an orientation on an n-dimensional complex K is given by a nondegenerate simplicial map $\xi : K \to \Delta^n$ to the n-simplex. In this case each i-skeleton $K^{(i)}$ of K also will be given an orientation by means of ξ.

On the product of two ordered sets $A \times B$ we define the *product order* by the rule: $(a, b) < (a', b')$ if either $a < a'$ and $b \leq b'$ or $a \leq a'$ and $b < b'$. Let σ be an oriented k-simplex $x_0, \ldots, x_k$. The product order on $\{x_0, \ldots, x_k\} \times \{0, 1\}$ defines a triangulation on $\sigma \times [0, 1]$ by the following rule: every chain $(x_{i_0}, y_{j_0}) < \ldots < (x_{i_r}, y_{j_r})$, $y_j = 0, 1$, spans a simplex. We note that the set of vertices for this triangulation equals $\{x_0, \ldots, x_k\} \times \{0, 1\}$.

Proposition 4.2.1 *For every simplicial map $f : X \to Y$ of an oriented simplicial complex X, the mapping cylinder M_f admits an oriented triangulation with the set of vertices equal to the disjoint union of vertices of X and Y.*

Proof: We note that every characteristic map $q|_{\sigma^k \times I} : \sigma^k \times [0, 1] \to q(\sigma^k \times [0, 1])$ is simplicial with respect to the above triangulation on $\sigma^k \times [0, 1]$. This means that every cell in M_f is a simplicial complex. Since the orientations on simplices σ agree with each other, the union of closed cells $q_{\sigma^k \times I}(\sigma^k \times I)$ together with Y forms a simplicial complex. ∎

Proposition 4.2.2 *For every n there is a constant c_n so that the quotient map of the product $X \times [0,1]$ to the mapping cylinder M_g of any simplicial map $g : X \to Y$ of oriented n-dimensional simplicial complexes with the uniform metric is c_n-Lipschitz.*

Proof: For each $k > 0$, by a k-prism we mean a subset of $X \times I$ of the form $\sigma \times I$ for some $k-1$-simplex σ of X. First, for each k, there exists a constant λ_k so that $q : X \times I \to M_g$ restricted to each k-prism is λ_k-Lipschitz.

With the λ_k as above, take $c_n = \max\{\lambda_k \mid 1 \leq k \leq 2n+1\}$. It remains to show that q is c_n-Lipschitz. To this end, let x and y in $X \times I$ be given. Take a simplex σ in X of minimal dimension with $x \in \sigma \times I$, and a simplex $\tau \in X$ of minimal dimension with $y \in \tau \times I$.

Let i denote the inclusion of $X \times I$ into $l_2 \times I$, and j the inclusion of the uniform complex $(X \times I)_U$ into $(l_2 \times I)_U$. Let $\bar{q}$ denote the extension of q to $l_2 \times I$ given by defining $\bar{q} = q$ on vertices, and extending by linearity to all of l_2.

Let ρ be a simplex in l_2 of minimal dimension so that $i(\sigma)$ and $i(\tau)$ are faces of ρ. Then, $\dim \rho \leq \dim \sigma + \dim \tau + 1 \leq 2n + 1$. Now, $\bar{q}$ is c_n-Lipschitz when restricted to the prism $\rho \times I$, and $j^{-1}\bar{q}i(\sigma) = q(\sigma)$, where j^{-1} and i are isometries. Similarly, $j^{-1}\bar{q}i(\tau) = q(\tau)$. Thus, $d(q(x), q(y)) = d(j^{-1}\bar{q}i(x), j^{-1}\bar{q}i(y)) \leq c_n d(x,y)$. Thus, q is c_n-Lipschitz. ∎

Proposition 4.2.3 *Let $A \subset W \subset X$ be subsets in a geodesic metric space X such that the r-neighborhood $N_r(A)$ is contained in W and let $f : W \to Y$ be a continuous map to a metric space Y. Assume that the restrictions $f|_{N_r(A)}$ and $f|_{W \setminus N_r(A)}$ are ϵ-Lipschitz. Then f is ϵ-Lipschitz.*

Proof: Let $x, y \in W$ be two points. The only possibility for difficulties with checking the Lipschitz condition can occur when x and y lie in different sets $N_r(A)$ and $W \setminus N_r(A)$. Let $x \in N_r(A)$ and $y \in W \setminus N_r(A)$ and let $\gamma : [0, d(x,y)] \to X$ be a geodesic segment joining them in X. Then there is a $t \in [0, d(x,y)]$ such that $\gamma(t) \in \partial N_r(A)$, where $\partial N_r(A) = \{z \in X \mid d(z,A) = r\}$. Then $d_Y(f(x), f(y)) \leq d_Y((x), f(\gamma(t))) + d_Y(f(\gamma(t), f(y)) \leq \epsilon d_X(x, \gamma(t)) + \epsilon d_X(\gamma(t), y) = \epsilon d_X(x,y)$. ■

We recall that $L(\mathcal{U})$ denotes the Lebesgue number of a cover $\mathcal{U}$ of a metric space. If $\mathcal{U}$ is a cover of a subset $A \subset X$ of a metric space, we let $b(\mathcal{U})$ denote the least upper bound for the diameters of $U \in \mathcal{U}$ and $\nu(\epsilon, k)$ denote the number defined by Proposition 4.1.1.

Lemma 4.2.4 *Let $r > 8\epsilon$ and let $\mathcal{V}$ and $\mathcal{U}$ be covers of the r-neighborhood $N_r(A)$ of a closed subset A in a geodesic metric space X by open sets such that both $\mathcal{U}$ and $\mathcal{V}$ have order $\leq n+1$, $N(\mathcal{V})$ is orientable, and $L(\mathcal{U}) > b(\mathcal{V}) > L(\mathcal{V}) \geq \nu(\epsilon/4c_n, n)$, where c_n is the constant from* Proposition 4.2.2. *Then there is an ϵ-Lipschitz map $f : N_r(A) \to M_g$ to the uniform mapping cylinder of a simplicial map $g : N(\mathcal{V}) \to N(\mathcal{U})$ between the nerves such that f is uniformly cobounded, $f|_{\partial N_r(A)} = q(p_{\mathcal{V}}|_{\partial N_r(A)}, 0)$, and $f|_A = p_{\mathcal{U}}|_A$, where $p_{\mathcal{U}} : N_r(A) \to N(\mathcal{U})$ and $p_{\mathcal{V}} : N_r(A) \to N(\mathcal{V})$ are the canonical projections to the nerves.*

Proof: We recall that the simplicial complexes $N(\mathcal{U})$ and $N(\mathcal{V})$ inherit their metrics from the Hilbert spaces $l_2(\mathcal{U})$ and $l_2(\mathcal{V})$. By Proposition 4.1.1 the projections $p_{\mathcal{U}} : N_r(A) \to N(\mathcal{U})$ and $p_{\mathcal{V}} : N_r(A) \to N(\mathcal{V})$ are $\epsilon/4c_n$-Lipschitz. Let $t_x = \frac{2d_X(x,A)}{c_n r}$, we define

$$f(x) = \begin{cases} q(p_{\mathcal{V}}(x), 2 - t_x) & \text{if } d(x,A) > c_n r/2 \\ t_x g p_{\mathcal{V}}(x) + (1 - t_x) p_{\mathcal{U}}(x) & \text{otherwise} \end{cases}$$

In the first case $q^{-1}f(x)$ lies in $N(\mathcal{V}) \times [0,1)$. The map $q^{-1}f : N_r(A) \to N(\mathcal{V}) \times [0,1)$ is $\max\{\epsilon\sqrt{2}/4c_n . 2\sqrt{2}/c_n r\}$-Lipschitz and is therefore ϵ/c_n-Lipschitz. In the second case the linear combination is taken in the Hilbert space $l_2(\mathcal{U})$. It belongs to $N(\mathcal{U})$, since $gp_{\mathcal{V}}(x)$ and $p_{\mathcal{U}}(x)$ lie in the simplex of $N(\mathcal{U})$ spanned by all $U \in \mathcal{U}$ that contain x. Note that the coefficients in the linear combination are $2/c_n r$-Lipschitz functions. Therefore, they are $\epsilon/4c_n$-Lipschitz, as $2/r \leq \epsilon/4$.

Proposition 4.2.3 implies q is c_n-Lipschitz, so the restriction $qq^{-1}f|_{N_{c_n r}(A)\setminus N_{c_n r/2}}$ is ϵ-Lipschitz. Since the coefficients t_x and $1-t_x$ are $\epsilon/4c_n$-Lipschitz and $\|gp_{\mathcal{V}}(x)\|$ and $\|p_{\mathcal{U}}(x)\| \leq 1$, it follows from the Leibnitz rule that the restriction of f to $N_{c_n r/2}(A)$ is also ϵ-Lipschitz. In view of Proposition 4.2.3 we can conclude that f is an ϵ-Lipschitz map. Note that $f|_{\partial N_r(A)} = p_{\mathcal{V}}|_{\partial N_r(A)}$ and $f|_A = p_{\mathcal{U}}|_A$.

Finally, we show $\operatorname{diam}(f^{-1}(q(\sigma \times I))) \leq 2b(\mathcal{U})$. For x and y in $f^{-1}(q(\sigma \times I))$, there are three cases to consider, based on $\operatorname{dist}(x, A)$ and $\operatorname{dist}(y, A)$. If both $\operatorname{dist}(x, A)$ and $\operatorname{dist}(y, A) > c_n r/2$, then $f(x) = q(p_{\mathcal{V}}(x), 2 - t_x)$ and $f(y) = q(p_{\mathcal{V}}(y), 2 - t_y)$, so x and y are in σ under the projection to the nerve $p_{\mathcal{V}}$. So, there is some set $V \in \mathcal{V}$ containing x and y. Thus, $\operatorname{dist}(x, y) \leq b(\mathcal{V}) < b(\mathcal{U})$. Next, if $\operatorname{dist}(x, A) < c_n r/2$ and $\operatorname{dist}(y, A) \geq c_n r/2$, then $f(x)$ is the same as above, and $f(y)$ is a linear combination of elements of a simplex in $N(\mathcal{V})$, with coefficients at most 1. The map g takes an open set $V \in \mathcal{V}$ containing x into a set $U \in \mathcal{U}$ containing the elements of the linear combination in the expression for $f(y)$. Thus, $\operatorname{dist}(x, y) \leq 2b(\mathcal{U})$. Finally, if both elements lie in the closed $c_n r/2$ neighborhood of A, then both of the elements lie in a linear combination of elements of a simplex in $N(\mathcal{U})$, so $\operatorname{dist}(x, y) \leq 2b(\mathcal{U})$. ■

4.3 Groups Acting on Trees

Assume that a group Γ acts on a metric space X. The following theorem is analogous to Theorem 3.2.1. By taking X to be a tree instead of an arbitrary finite dimensional metric space, we are able to give a sharp estimate for the upper bound of the dimension.

Lemma 4.3.1 *Assume that a finitely generated group* Γ *acts by isometries on a tree* X *with a base point* x_0. *Suppose that* $\text{asdim} W_R(x_0) \leq k$ *for all* R. *Then* $\text{asdim}\Gamma \leq k+1$.

Proof: Let $S = S^{-1}$ be a finite generating set for Γ. We define a map $\pi : \Gamma \to X$ by the formula $\pi(g) = g(x_0)$. Using the uniqueness of a geodesic segment between any two points in X this map can be canonically extended to a Γ-equivariant map $\bar{\pi} : Y \to X$ of the Cayley graph Y of the group Γ. Let $\lambda = \max\{d_X(s(x_0), x_0) \mid s \in S\}$. We show now that π is λ-Lipschitz. Since the metric d_S on Γ is induced from the geodesic metric on the Cayley graph, it suffices to check that $d_X(\pi(g), \pi(g')) \leq \lambda$ for all $g, g' \in \Gamma$ with $d_S(g, g') = 1$. Without loss of generality we assume that $g' = gs$, where $s \in S$. Then $d_X(\pi(g), \pi(g')) = d_X(g(x_0), gs(x_0)) = d_X(x_0, s(x_0)) \leq \lambda$. Since every edge in the Cayley graph maps via $\bar{\pi}$ to a geodesic segment, it follows that $\bar{\pi}$ is also λ-Lipschitz.

Observe that for all $R > 0$, $\gamma \in \Gamma$, and $x \in X$ we have $W_R(x_0) = \pi^{-1}(B_R(x_0))$, $\gamma B_R(x) = B_R(\gamma(x))$ and $\gamma(\bar{\pi}^{-1}(B_R(x))) = \bar{\pi}^{-1}(B_R(\gamma(x)))$. We note that the spaces $W_R(x_0)$ and $\bar{\pi}^{-1}(B_R(x_0))$ are in a finite Hausdorff distance from each other and therefore, coarsely equivalent. We denote the latter by $\bar{W}_R(x_0)$.

Let c_k be the Lipschitz constant defined by uniformization of the product $K' \times [0,1]$, for a k-dimensional simplicial complex K' (see Proposition 4.2.2).

Given a (small) number $\epsilon > 0$, we will construct an ϵ-Lipschitz, uniformly cobounded map $\psi : Y \to K$ to a uniform $k+1$-dimensional simplicial complex. Let $\nu = \nu(\epsilon/4c_k, k)$ be from Proposition 4.1.1, and take $r > \max\{\nu, 8/\epsilon\}$. Since $\text{asdim} X \leq 1$, there is a uniformly bounded cover $\mathcal{W}$ of X with multiplicity 2 such that the λr-enlargement $\{N_{\lambda r}(W)\}$ is a cover of X with multiplicity 2. Indeed, using Gromov's first definition of asdim, we can find two families of uniformly bounded, d-disjoint sets $\mathcal{W}_1$ and $\mathcal{W}_2$ whose union covers X with $d \gg \lambda r$; then, put $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$. For each $W \in \mathcal{W}$, let $\bar{W}$ denote $\bar{\pi}^{-1}(W)$. As $\mathcal{W}$ is uniformly bounded, there is some $R > 0$ so that for each $W \in \mathcal{W}$ we can find $x_W \in X$ with the property that $N_{\lambda r}(W) \subset B_R(x_W)$. As $\bar{\pi}$ is λ-Lipschitz, we have $\bar{\pi}(N_r(\bar{W})) \subset N_{\lambda r}(W)$. Thus, $N_r(\bar{W}) \subset \bar{\pi}^{-1}(B_R(x_W)) = \bar{W}_R(x_W)$. Let $\gamma_W \in \Gamma$ have $\pi(\gamma_W) = x_W$. We use the condition asdim$\bar{W}_R(x_0) \leq k$ to construct two families of uniformly bounded covers $\mathcal{V}$ and $\mathcal{U}$ of the set $\bar{W}_R(x_0)$ both with multiplicity $\leq k+1$ such that the Lebesgue number $L(\mathcal{V})$ is greater than r and $L(\mathcal{U}) > b(\mathcal{V})$, where $b(\mathcal{V})$ is an upper bound for diameters of the cover $\mathcal{V}$.

First, we construct a family of uniformly cobounded ϵ-Lipschitz maps $\phi_W : N_r(\bar{W}) \to K_W$ to uniform k-dimensional simplicial complexes as the canonical projections to the nerve of $\gamma_W \mathcal{V}$ restricted to $N_r(\bar{W})$.

Next, let W and W' be elements of $\mathcal{W}$ such that $W \cap W' \neq \emptyset$. Thus, in the nerve, $N(\mathcal{W})$, the sets W and W' define an edge e. Put $A_e = W \cap W'$, and let $\bar{A}_e$ denote $\bar{\pi}^{-1}(A_e)$. Observe that $N_r(\bar{A}_e) \subset N_r(\bar{W})$ and $N_r(\bar{W}')$. Let $\mathcal{V}_e$ and $\mathcal{U}_e$ denote the restrictions of the covers $\mathcal{V}$ and $\mathcal{U}$ to $N_r(\bar{A}_e)$. Without loss of generality we may assume that the nerve $N(\mathcal{V})$ is orientable. Then the nerve $N(\mathcal{V}_e)$ is orientable, and we may take the triangulation on the mapping cylinder M_{g_e} of a simplicial map $g_e : N(\mathcal{V}_e) \to N(\mathcal{U}_e)$ as in Proposition 4.2.1, and consider M_{g_e} as a uniform simplicial complex. As $r > 8/\epsilon$, we may apply Lemma 4.2.4 to $\bar{A}_e \subset Y$ and to the covers $\mathcal{V}_e$ and $\mathcal{U}_e$ to obtain an ϵ-Lipschitz map $f_e : N_r(\bar{A}_e) \to M_{g_e}$. We note that the $N_r(\bar{A}_e)$ are disjoint for distinct edges in the nerve. Indeed, $\bar{\pi}(N_r(\bar{A}_e)) \subset N_{\lambda r}(A_e) \subset N_{\lambda r}(W) \cap N_{\lambda r}(W')$; so, if $N_r(\bar{A}_e) \cap N_r(A_{e'}) \neq \emptyset$, then the λr-enlargement of $\mathcal{W}$ would have multiplicity at least 3. Thus, for each $W \in \mathcal{W}$ define $\psi_W : N_r(\bar{W}) \to K_W \cup_{W \in e} M_{g_e} = L_W$ to the uniform complex L_W, with mapping cylinders attached as the union of the map $\phi_W|_{N_r(\bar{W}) \setminus \cup_{W \in e} N_r(\bar{A}_e)}$ and the restrictions $f_e|_{N_r(\bar{A}_e) \cap N_r(\bar{W})}$, for all edges e in $N(\mathcal{W})$ which contain W as a vertex in the nerve $N(\mathcal{W})$.

We construct K by gluing together the L_W. Clearly, the dimension of K is at most $k+1$. The maps $\psi_W : Y \to K$ agree on the common parts $\bar{A}_e$ so they define a map $\psi : Y \to K$. The map ψ is ϵ-Lipschitz by Proposition 4.2.3, and uniformly cobounded by the Lemma 4.2.4. ■

We can improve on the results from Chapter 3 using the Bass-Serre classification theorem. In particular, it is known that every group acting without inversions on a tree with compact quotient is the fundamental group of a graph of groups as described in Chapter 2. Thus, we have:

Theorem 4.3.2 *Let π be the fundamental group of a graph of groups where each vertex group has* asdim *not exceeding n. Then,* $\mathrm{asdim}\pi \leq n+1$.

Corollary 4.3.3 *If* $\mathrm{asdim}A \leq n$ *and* $\mathrm{asdim}B \leq n$, *then* $\mathrm{asdim}A *_C B \leq n+1$ *and* $\mathrm{asdim}A*_C \leq n+1$.

This upper bound is tight for the case of the HNN extension. It is tight for some cases of the free product with amalgamation. We believe that the specific upper bound for the amalgamated free product should be $\mathrm{asdim}A *_C B \leq \max\{\mathrm{asdim}A, \mathrm{asdim}B, \mathrm{asdim}C + 1\}$. It seems that the proof of this requires techniques that are more combinatorial in nature.

CHAPTER 5
PROPERTY A FOR GROUPS ACTING ON FINITE DIMENSIONAL SPACES

5.1 Mappings to Nerves in l_1

Let $\mathcal{U}$ be an open cover of the metric space X with finite multiplicity. Let $p_{\mathcal{U}} : X \to N(\mathcal{U})$ denote the projection to the nerve of $\mathcal{U}$ defined by the partition of unity $\{\phi_U : X \to \mathbb{R}\}_{U \in \mathcal{U}}$ where $\phi_U(x) = d(x, X \setminus U)/(\sum_{V \in \mathcal{U}} d(x, X \setminus V))$. Realize $N(\mathcal{U})$ in $l_1(\mathcal{U})$ by sending each vertex U to an element of an orthonormal basis. Endow $N(\mathcal{U})$ with the metric it inherits as a subset of $l_1(\mathcal{U})$

Proposition 5.1.1 (Similar to Proposition 4.1.1) *For every $k > 0$ and for every $\epsilon > 0$ there exists an $L > 0$ such that for every open cover of X with Lebesgue number $L(\mathcal{U}) \geq L$ and multiplicity $\leq k + 1$, the projection $p_{\mathcal{U}} : X \to N(\mathcal{U})$ is ϵ-Lipschitz.*

Proof. Take L greater than $(2k + 3)^2/\epsilon$. Let $x, y \in X$ and $U \in \mathcal{U}$. The triangle inequality implies $|d(x, X \setminus U) - d(y, X \setminus U)| \leq d(x, y)$. Then

$$\begin{aligned}
|\phi_U(x) - \phi_U(y)| \quad \leq \quad & \frac{d(x,y)}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)} + \\
& + d(y, X \setminus U)\Big|\frac{1}{\sum_{V \in \mathcal{U}} d(x, X \setminus V)} - \frac{1}{\sum_{V \in \mathcal{U}} d(y, X \setminus V)}\Big| \\
& \leq \frac{1}{L(\mathcal{U})} d(x,y) + \frac{1}{L(\mathcal{U})}\Big(\sum_{V \in \mathcal{U}} |d(x, X \setminus V) - d(y, X \setminus V)|\Big) \\
& \leq \frac{2k+3}{L} d(x,y).
\end{aligned}$$

Then

$$\begin{aligned}
\|p(x) - p(y)\|_1 \quad &= \quad \sum_{U \in \mathcal{U}} |\phi_U(x) - \phi_U(y)| \\
&\leq \quad (2k+2)\left(\frac{2k+3}{L} d(x,y)\right) \leq \frac{(2k+3)^2}{L} d(x,y) \leq \epsilon d(x,y). \blacksquare
\end{aligned}$$

5.2 Property A: Definition and Reformulations

Recall that a space X is said to have property A if there exists a family of functions

$$a^n : X \to \text{Prob}(X)$$

satisfying:

1. for every $n > 0$ there is an $R = R(n) > 0$ such that
$$\text{supp}(a^n_x) \subset B_R(x)$$
for all $x \in X$; and
2. for every $K > 0$,
$$\lim_{n\to\infty} \sup_{d(z,w)<K} \|a^n_z - a^n_w\|_1 = 0.$$

By way of notation, let $X \in A(R(n), N(K, \epsilon))$ indicate that the metric space X has property A, and that there exists a family of maps $a^n : X \to \text{Prob}(X)$ satisfying the following conditions:

1. for every $n > 0$ there is an $R = R(n)$ for which $\text{supp}(a^n_x) \subset B_R(x)$, for every $x \in X$ and
2. for every $K > 0$ and for every $\epsilon > 0$ there is an $N = N(K, \epsilon)$ so that if $n \geq N$,
$$\sup_{d(x,x')<K} \|a^n_x - a^n_{x'}\|_1 < \epsilon.$$

The following proposition was collected from results of Tu [17]. The statement that subsets inherit property A is not as important as the estimates on the supports and the norms.

Proposition 5.2.1 *Suppose that* $X \subset Y$ *where* $Y \in A(R, N)$. *Then* $X \in A(3R, N)$.

Proof: (Collected from results of Tu [17].) Let $b^n : Y \to \mathrm{Prob}(Y)$ be a family of maps such that for every $n > 0$ there is an $R > 0$ such that $\mathrm{supp}(b^n_y) \subset B_R(y)$ for all $y \in Y$, and such that for any $K > 0$ and for any $\epsilon > 0$ there exists an $N > 0$ such that if $n \geq N$, then $\sup_{d(y,y')<K} \|b^n_y - b^n_{y'}\| < \epsilon$. For each $y \in Y$, let $p(y) \in X$ be a point such that $d(y, p(y)) \leq 2d(y, X)$. Let $V : \mathrm{Prob}(Y) \to \mathrm{Prob}(X \times Y)$ be defined by

$$V(\eta)(x,y) = \begin{cases} \eta(y), & \text{if } x = p(y) \\ 0 & \text{otherwise} \end{cases}$$

Define $a^n : X \to \mathrm{Prob}(X \times Y)$ by $a^n_x(\xi, \eta) = V(b^n_x)(\xi, \eta)$.

First, notice that

$$\|a^n_x\|_{l_1(X\times Y)} = \sum_{(\xi,\eta)} |V(b^n_x)(\xi,\eta)| = \sum_{\eta} |b^n_x(\eta)| = \|b^n_x\|_{l_1(Y)} = 1.$$

Next,

$$\begin{aligned} \|a^n_x - a^n_{x'}\|_{l_1(X\times Y)} &= \sum_{(\xi,\eta)} |a^n_x(\xi,\eta) - a^n_{x'}(\xi,\eta)| \\ &= \sum_{(\xi,\eta)} |V(b^n_x)(\xi,\eta) - V(b^n_{x'})(\xi,\eta)| \\ &= \sum_{\eta} |b^n_x(\eta) - b^n_{x'}(\eta)| = \|b^n_x - b^n_{x'}\|_{l_1(Y)}. \end{aligned}$$

Finally, define $c^n : X \to \mathrm{Prob}(X)$ by $c^n_x(\xi) = \|a^n_x(\xi, \cdot)\|_{l_1(Y)}$. Observe first that $\|c^n_x\|_{l_1(X)} = \sum_\xi |c^n_x(\xi)| = \sum_\xi \|a^n_x(\xi,\cdot)\|_{l_1(Y)} = \|a^n_x\|_{l_1(X\times Y)} = 1$. Then,

$$\begin{aligned} \|c^n_x - c^n_{x'}\|_{l_1(X)} &= \sum_{\xi\in X} \big| \|a^n_x(\xi,\cdot)\|_{l_1(Y)} - \|a^n_{x'}(\xi,\cdot)\|_{l_1(Y)} \big| \\ &\leq \sum_{\xi\in X} \|a^n_x(\xi,\cdot) - a^n_{x'}(\xi,\cdot)\|_{l_1(Y)} \\ &= \|a^n_x - a^n_{x'}\|_{l_1(X\times Y)} = \|b^n_x - b^n_{x'}\|. \end{aligned}$$

Thus, we need only show that the support condition holds. To this end, suppose $\xi \in \mathrm{supp}(c^n_x)$. Then $\|a^n_x(\xi,\cdot)\|_{l_1(Y)} > 0$. Since $\mathrm{supp}(a^n_x) \subset p(\mathrm{supp}(b^n_x)) \times Y \subset p((B_R(x)) \times Y \subset B_{3R}(x) \times Y$, we conclude that $\xi \in B_{3R}(x)$. ∎

The following lemma allows for the introduction of an additional parameter in the definition of the functions yielding property A.

Lemma 5.2.2 *Suppose Z is a metric space and $\{a^{m,l}\}_{m,l}$ is a family of functions*

$$a^{m,l} : Z \to \mathrm{Prob}(Z)$$

such that

1. *for every pair (m, l) there is an $R = R(m, l) > 0$ such that*

$$\mathrm{supp}(a_z^{m,l}) \subset B_R(z),$$

for every $z \in Z$; and

2. *for every $\eta > 0$, and for every $D > 0$ there exists a pair (M, L) such that*

$$\sup_{d(z,w)<D} \|a_z^{m,L} - a_w^{m,L}\|_1 < \eta,$$

for all $m \geq M$.

Then Z has property A.

Proof: We propose to construct a family of functions $a^n : Z \to \mathrm{Prob}(Z)$ such that for every n, there exists an R such that $\mathrm{supp}(a_z^n) \subset B_R(z)$, for every $z \in Z$, and such that for every $K > 0$ and every $\epsilon > 0$ there exists an N such that $\sup\{\|a_z^n - a_w^n\|_1 \mid d(z,w) < K\} < \epsilon$, for all $n \geq N$.

For each $n > 0$, let (M_n, L_n) denote the pair from (2) for $\eta = \frac{1}{n}$ and $D = n$. Define the family $a^n : Z \to \mathrm{Prob}(Z)$ by $a_z^n = a_z^{M_n,L_n}$. We claim that this family satisfies the hypotheses for property A outlined in the preceding paragraph.

First, we have

$$\|a_z^n\|_1 = \sum_{w\in Z} |a_z^n(w)| = \sum_{w\in Z} |a_z^{M_n,L_n}(w)| = 1,$$

as $a_z^{M_n,L_n} \in \mathrm{Prob}(Z)$. Thus, $a_z^n \in \mathrm{Prob}(Z)$.

Next, for each n take $R = R(M_n, L_n)$. Then, for every $z \in Z$, we have $\mathrm{supp}(a_z^n) = \mathrm{supp}(a_z^{M_n,L_n}) \subset B_R(z)$.

Finally, let $\epsilon > 0$ and $K > 0$ be given. Take $N \in \mathbb{Z}^+$ so large that $N \geq K$ and $\frac{1}{N} < \epsilon$. Then, for every $n \geq N$,

$$\begin{aligned}\sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 &\leq \sup_{d(z,w)<n} \|a_z^n - a_w^n\|_1 \\ &= \sup_{d(z,w)<n} \|a_z^{M_n,L_n} - a_w^{M_n,L_n}\|_1 \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.\end{aligned}$$

Thus, the family $\{a^n\}$ yields property A for Z. ■

5.3 Groups Acting on Finite Dimensional Spaces

In the following theorem we use the obvious fact that if X and Y are isometric metric spaces, then $X \in A(R(n), N(K, \epsilon))$ if and only if $Y \in A(R(n), N(K, \epsilon))$.

The following theorem is the analog of Theorem 3.2.1 for property A.

Theorem 5.3.1 *Let Γ be a finitely generated group acting on the metric space X by isometries. Suppose that* $\mathrm{asdim} X \leq k$ *and that for every $d > 0$, $W_d(x_0)$ has property A. Then, Γ has property A.*

Proof: Define $\pi : \Gamma \to X$ by $\pi(g) = gx_0$. Let $S = S^{-1}$ be a finite set of generators for Γ. Let $\lambda = \max\{d(x_0, s(x_0)) \mid s \in S\}$. We show now that π is λ-Lipschitz. Since the metric d_S on Γ is induced from the geodesic metric on the Cayley graph, it suffices to check that $d_X(\pi(g), \pi(g')) \leq \lambda$ for all $g, g' \in \Gamma$ with $d_S(g, g') = 1$. Without loss of generality we assume that $g' = gs$, where $s \in S$. Then $d_X(\pi(g), \pi(g')) = d_X(g(x_0), gs(x_0)) = d_X(x_0, s(x_0)) \leq \lambda$.

Let $L > 0$ be given and take a cover $\mathcal{U} = \{U_i\}_i$ of X with Lebesgue number $L(\mathcal{U}) = L$ and multiplicity $\leq k+1$, such that the L-enlargement $\mathcal{V} = \{V_i\}$ of $\mathcal{U}$ is also a cover of multiplicity $k+1$. To see that this can be done, take uniformly bounded, $3L$-disjoint families $\mathcal{U}_0, \ldots, \mathcal{U}_k$ whose union covers X, and put $\mathcal{U} = \cup_i \mathcal{U}_i$.

Next, take $d > 0$ sufficiently large and $x_i \in X$ such that $V_i \subseteq B_d(x_i)$, for each i. Finally, take $\gamma_i \in \Gamma$ with $\gamma_i x_0 = x_i$.

We have $W_d(x_0) = \pi^{-1}(B_d(x_0))$, and $\gamma\pi^{-1}(B_d(x_0)) = \pi^{-1}(B_d(\gamma x_0))$. Next, since $\pi^{-1}(B_d(\gamma x_0))$ is isometric to $\pi^{-1}(B_d(x_0))$, and as $W_d(x_0) \in A(R(n), N(K,\epsilon))$, we have $\pi^{-1}(B_d(\gamma_i x_0)) \in A(R(n), N(K,\epsilon))$. Since $\pi^{-1}(V_i) \subset \pi^{-1}(B_d(\gamma_i x_0))$, by Proposition 5.2.1, we conclude that each $\pi^{-1}(V_i) \in A(3R(n), N(K,\epsilon))$.

Let $c^{n,L} : W_d(x_0) \to \mathrm{Prob}(W_d(x_0))$ be maps giving $W_d(x_0) \in A(R(n), N(K,\epsilon))$. Thus, we can take maps $(\gamma_i b^{n,L}) : \pi^{-1}(V_i) \to \mathrm{Prob}(\pi^{-1}(V_i))$ yielding property A for the $\pi^{-1}(V_i)$ in such a way that $\|(\gamma_i b^{n,L}_x) - (\gamma_i b^{n,L}_{x'})\|_1 \leq \|c^{n,L}_x - c^{n,L}_{x'}\|_1$, for x and x' in $\pi^{-1}(\gamma_i(B_d(x_0)))$. Extend these maps to $\mathrm{Prob}(\Gamma)$ by defining $(\gamma_i b^{n,L}_g)(h) = 0$ whenever $h \in \Gamma \setminus \pi^{-1}(V_i)$. For notational ease, let b^i denote $(\gamma_i b^{n,L})$, when n and L are understood.

Let $p_{\mathcal{U}} : X \to N(\mathcal{U})$ denote the canonical projection to the nerve of $\mathcal{U}$ in $l_1(\mathcal{U})$. For each $g \in \Gamma$, let

$$p_{\mathcal{U}}(\pi(g)) = \sum_{U_i \in \mathcal{U}} t_i^g U_i$$

be a presentation of the image of gx_0 in the nerve $N(\mathcal{U})$ in barycentric coordinates. Thus, for each $g \in \Gamma$, $t_i^g \geq 0$, for all $U_i \in \mathcal{U}$, and $\sum_i t_i^g = 1$. Also, since $\mathcal{U}$ has multiplicity $\leq k+1$, for any fixed $g \in \Gamma$, at most $k+1$ of the t_i^g are nonzero. Note, too that $g \mapsto p_{\mathcal{U}}(\pi(g))$ is $(\lambda(2k+3)^2/L)$-Lipschitz, by Proposition 5.1.1.

Define $a^{n,L} : \Gamma \to \mathrm{Prob}(\Gamma)$ by

$$a_g^{n,L}(h) = \sum_{U_i \in \mathcal{U}} t_i^g b_g^i(h).$$

We claim that these families yield property A for Γ.

First, we compute:

$$\|a_g^{n,L}\|_1 = \sum_{h\in\Gamma} |a_g^{n,L}(h)| = \sum_{h\in\Gamma} |\sum_{g\in U_i} t_i^g b_g^i(h)| = \sum_{g\in U_i} t_i^g \sum_{h\in\Gamma} b_g^i(h) = \sum_{g\in U_i} t_i^g = 1.$$

Next, given n and L, take R so that $\mathrm{supp}(c_g^{n,L}) \subset B_R(g)$ for every $g \in W_d(x_0)$. Then, $\mathrm{supp}(a_g^{n,L}) \subset \cup_i \mathrm{supp}(b_g^i) \subset B_{3R}(g)$ for each g by Proposition 5.2.1.

Finally, let $\epsilon > 0$ and $K > 0$ be given. By Lemma 5.2.2, we only have to prove that there exists a pair (N, L) such that if $n \geq N$ and $d(z, w) < K$, then $\|a_z^{n,L} - a_w^{n,L}\|_1 < \epsilon$. Choose $L \gg K$ and so large that $\frac{\lambda K(2k+3)^2}{L} < \epsilon/2$. Next, choose N so large that $(k+1)\|c_z^{n,L} - c_w^{n,L}\|_1 \leq \epsilon/2$, if $n \geq N$ and $d(z, w) < K$ in $\pi^{-1}(B_d(x_0))$. Then, if $d(z, w) < K$ in some V_i then we also have $(k+1)\|b_z^i - b_w^i\|_1 \leq \epsilon/2$, by Proposition 5.2.1.

Now, consider any z and w in Γ with $d(z, w) < K$. In the nerve, we have $z = \sum t_i^z U_i$ and $w = \sum t_j^w U_j$. Next, suppose some $\pi^{-1}(V_i)$ with $t_i^z > 0$ fails to contain w. Then, as $z \in \pi^{-1}(U_i)$, as $K < L$ and as $d(z, w) < K$, w is in $\pi^{-1}(N_L(U_i)) = \pi^{-1}(V_i)$. Thus, w is in all V_i with $t_i^z > 0$. The argument is symmetric in z and w, so $\{U_i\} = \{U_j\}$.

So, we have:

$$
\begin{aligned}
\|a_z^{n,L} - a_w^{n,L}\|_1 &= \sum_{g\in\Gamma}\Big|\sum_{U_i\in\mathcal{U}} t_i^z b_z^i(g) - t_i^w b_w^i(g)\Big| \\
&\leq \sum_{g\in\Gamma}\sum_{U_i\in\mathcal{U}}\Big|t_i^z b_z^i(g) - t_i^w b_w^i(g)\Big| \\
&= \sum_{g\in\Gamma}\sum_{U_i\in\mathcal{U}}\Big|(t_i^z - t_i^w)b_z^i(g) + t_i^w(b_z^i(g) - b_w^i(g))\Big| \\
&\leq \sum_{g\in\Gamma}\sum_{U_i\in\mathcal{U}}|(t_i^z - t_i^w)b_z^i(g)| + |t_i^w(b_z^i(g) - b_w^i(g))| \\
&= \sum_{U_i\in\mathcal{U}}\sum_{g\in\Gamma}|(t_i^z - t_i^w)b_z^i(g)| + |t_i^w(b_z^i(g) - b_w^i(g))| \\
&= \sum_{U_i\in\mathcal{U}}|t_i^z - t_i^w|\|b_z^i\|_1 + \sum_{U_i\in\mathcal{U}}|t_i^w|\|b_z^i - b_w^i\|_1 \\
&\leq \sum_{U_i\in\mathcal{U}}|t_i^z - t_i^w| + (k+1)\|c_z^{n,L} - c_w^{n,L}\|_1 \\
&\leq \frac{\lambda K(2k+3)^2}{L} + (k+1)\|c_z^{n,L} - c_w^{n,L}\|_1 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

So, $\sup_{d(z,w)<K}\|a_z^{n,L} - a_w^{n,L}\|_1 < \epsilon$. Thus, by Lemma 5.2.2, Γ has property A. ∎

5.4 Union Theorems for Property A

Recall that the subsets X and Y of the metric space Z are said to be an *excisive pair* if for every $R > 0$ there is an $S > 0$ so that

$$B_R(X) \cap B_R(Y) \subset B_S(X \cap Y).$$

In Tu [17] it is shown that if X and Y are an excisive pair and X and Y both have property A, then $X \cup Y$ has property A. In this section we show that property A is preserved by all finite unions and certain infinite unions. The following theorem is analogous to Theorem 3.1.1.

Theorem 5.4.1 *Let $X = \cup_\alpha X_\alpha$ be a metric space where each $X_\alpha \in A(R(n), N(K,\epsilon))$, for some fixed $R(n)$ and $N(K,\epsilon)$ independent of α. Suppose that for every $l > 0$ there is a $Y_l \subset X$, with property A such that the family $\{X_\alpha \setminus Y_l\}$ is l-disjoint. Then, X has property A.*

Proof: Since each $X_\alpha \in A(R(n), N(K,\epsilon))$ we can take $\xi^{n,\alpha} : X_\alpha \setminus Y_{7R(n)} \to \mathrm{Prob}(X_\alpha \setminus Y_{7R(n)})$ satisfying property A with $R = 3R(n)$, and $N = N(K,\epsilon)$, by Proposition 5.2.1. For ease of notation, put $l = l(n) = 7R(n)$.

Extend each $\xi_x^{n,\alpha}$ so that it is defined for each value of X by setting $\xi_x^{n,\alpha}(y) = 0$ if y is not in $X_\alpha \setminus Y_l$. Finally, define $\xi_x^{n,\alpha}$ to be identically zero for all x not in $X_\alpha \setminus Y_l$.

Since the family $\{X_\alpha \setminus Y_l\}$ is l-disjoint, we may define $\xi^n : X \setminus Y_l \to \mathrm{Prob}(X)$ by setting ξ_x^n equal to the unique $\xi_x^{n,\alpha}$ which is not identically zero. Notice that since $l = 7R(n)$, $\mathrm{supp}(\xi_x^{n,\alpha}) \cap \mathrm{supp}(\xi_{x'}^{n,\alpha'}) \subset B_{3R(n)}(x) \cap B_{3R(n)}(x') = \emptyset$, if $\alpha \neq \alpha'$.

Continue to take $l = 7R(n)$. Then, take functions $\eta^n : N_l(Y_l) \to \mathrm{Prob}(N_l(Y_l))$ satisfying the conditions for property A. Extend these functions to $\mathrm{Prob}(X)$ as before and define $\eta_x^n \equiv 0$ if $x \notin N_l(Y_l)$. Finally, define

$$t_Y^l(z) = \frac{d(z, X \setminus N_l(Y_l))}{d(z, Z \setminus N_l(Y_l)) + d(z, Y_l)},$$

and put $t^l_X(z) = 1 - t^l_Y(z)$. Then, the functions t^l are both $1/l$-Lipschitz. Indeed, for t^l_Y, we have

$$d(t^l_X(z), t^l_X(w)) = \left|\frac{d(z, X \setminus N_l(Y_l))}{d(z, X \setminus N_l(Y_l)) + d(z, Y_l)} - \frac{d(w, X \setminus N_l(Y_l))}{d(w, X \setminus N_l(Y_l)) + d(w, Y_l)}\right|$$

$$\leq \frac{1}{l}|d(z, X \setminus N_l(Y_l)) - d(w, X \setminus N_l(Y_l))| \leq \frac{1}{l}d(z, w).$$

Checking the condition for t^l_X is now easy.

Define $a^n : X \to \mathrm{Prob}(X)$ by

$$a^n_z = t^l_X(z)\xi^n_z + t^l_Y(z)\eta^n.$$

We now check the property A condition for this family of maps.

First, for a given $z \in X$, we find that

$$\begin{aligned} \|a^n_z\|_1 &= \|t^l_X(z)\xi^n_z + t^l_Y(z)\eta^n_z\|_1 \\ &= \sum_{w \in X} t^l_X(z)\xi^n_z(w) + t^l_Y(z)\eta^n_z(w) \\ &= t^l_X(z)\sum_{w \in X} \xi^n_z(w) + t^l_Y(z)\sum_{w \in X} \eta^n_z(w). \end{aligned}$$

If $z \in Y_l$, then $t^l_X(z) = 0$, and $\xi^n_z \equiv 0$. Then $\eta^n_z \not\equiv 0$, and $t^l_Y(z) = 1$, so

$$t^l_X(z)\sum_{w \in X} \xi^n_z(w) + t^l_Y(z)\sum_{w \in X} \eta^n_z(w) = 1.$$

The situation is similar if $z \in X \setminus N_l(Y_l)$. Finally, if $z \in N_l(Y_l) \setminus Y_l$, then

$$t^l_X(z)\sum_{w \in X} \xi^n_z(w) + t^l_Y(z)\sum_{w \in X} \eta^n_z(w) = t^l_X(z) + t^l_Y(z) = 1.$$

Thus, a^n defines a probability measure.

Next, for a given n take $R = \max\{3R(n), R(\eta^n)\}$. Then,

$$\mathrm{supp}(a^n_z) = \mathrm{supp}(\xi^n_z) \cup \mathrm{supp}(\eta^n_z) \subset B_{3R(n)}(z) \cup B_{R(\eta^n)}(z) = B_R(z),$$

for all $z \in X$.

Finally, we verify the last claim. To this end, let $\epsilon > 0$ and $K > 0$ be given. Notice that $R(n) \to \infty$ as $n \to \infty$, so we can take N so large that if $L = 7R(N)$, then $2K/L \leq \epsilon/2$, and if $n \geq N$, then $\sup_{d(z,w)<K} \|\xi^n_z - \xi^n_w\|_1 \leq \epsilon/4$ for all z and w in $X \setminus Y_L$, and $\sup_{d(z,w)<K} \|\eta^n_z - \eta^n_w\|_1 \leq \epsilon/4$ for all z and w in $N_L(Y_L)$. Then, for $n \geq N$, we have, for $d(z,w) < K$:

$$\|a^n_z - a^n_w\|_1 = \|t^L_X(z)\xi^n_z - t^L_X(w)\xi^n_w + t^L_Y(z)\eta^n_z - t^L_Y(w)\eta^n_w\|_1$$

There are various cases to consider based on where z and w lie. If both of z and w are in either $X \setminus Y_L$ or $N_L(Y)$, then there is nothing to check. Since $K < L$, it is not possible that z and w each lie in one of Y_L and $X \setminus N_L(Y_L)$. Suppose then that $z \in X \setminus N_L(Y_L)$ and $w \notin X \setminus N_L(Y_L)$. Then, $t^L_X(z) = 1$ and so we have:

$$\begin{aligned}
\|a^n_z - a^n_w\|_1 &= \|\xi^n_z - t^L_X(w)\xi^n_w - t^L_Y(w)\eta^n_w\|_1 \\
&\leq \|\xi^n_z - \xi^n_w + t^L_Y(w)(\xi^n_w - \eta^n_w)\|_1 \\
&\leq \|\xi^n - \xi^n_w\|_1 + |t^L_Y(w)|(\|\xi^n_w\|_1 + \|\eta^n_w\|_1) \\
&= \|\xi^n - \xi^n_w\|_1 + d(t^L_X(z), t^L_Y(w))(\|\xi^n_w\|_1 + \|\eta^n_w\|_1) \\
&\leq \|\xi^n - \xi^n_w\|_1 + (d(z,w)/L)(\|\xi^n_w\|_1 + \|\eta^n_w\|_1) \\
&\leq \epsilon/4 + 2K/L < \epsilon.
\end{aligned}$$

The situation is similar if $z \notin N_L(Y_L)$ and $w \in N_L(Y_L)$.

Finally, if both z and x are in $N_L(Y_L) \setminus Y_L$, then

$$\begin{aligned}
\|a^n_z - a^n_w\|_1 &= \|t^L_X(z)\xi^n_z - t^L_X(w)\xi^n_w + t^L_Y(z)\eta^n_z - t^L_Y(w)\eta^n_w\|_1 \\
&\leq (|t^L_X(z) - t^L_X(w)| + |t^L_Y(z) - t^L_Y(w)|) \\
&\quad + t^L_X(w)\|\xi^n_z - \xi^n_w\|_1 + t^L_Y(w)\|\eta^n_z - \eta^n_w\|_1 \\
&\leq K/L + K/L + \epsilon/4 + \epsilon/4 \\
&\leq 2K/L + \epsilon/2 < \epsilon.
\end{aligned}$$

So, in any case,

$$\sup_{d(z,w)<K} \|a_z^n - a_w^n\|_1 < \epsilon.$$

Thus, X has property A. ■

Corollary 5.4.2 *If X and Y have property A, then $X \cup Y$ has property A.*

Proof: Take the family $\{X_\alpha\}$ consisting of the sets X and Y. For each $l > 0$, put $Y_l = Y$. If $X \in A(R_X(n), N_X(K,\epsilon))$ and $Y \in A(R_Y(n), N_Y(K,\epsilon))$, then X and Y are both in $A(R(n), N(K,\epsilon))$, where $R = \max\{R_X, R_Y\}$ and $N = \max\{N_X, N_Y\}$. The l-disjointness of the family $\{X_\alpha \setminus Y_l\}$ is satisfied trivially. ■

5.5 Graphs of Groups

We now apply Theorems 5.3.1 and 5.4.1 to groups acting on trees by isometries to provide another proof of a result of Tu [17], namely, that the fundamental group of a graph of groups with each stablizer having property A also has property A. The following lemma is analogous to Lemma 3.3.1.

Lemma 5.5.1 *If a group Γ acts on a tree with compact quotient and finitely generated stabilizers Γ_x each with property A, for all vertices x, then $W_R(x_0)$ has property A for all R.*

Proof: Consider the subset $K \subset F(G,Y)$ of all words whose associated path c has $i(c) = G_{P_0}$. This set acts on the tree by left multiplication, and the fundamental group is a subset of K. We consider the R-stabilizer $W_R(P_0)$ of the fixed vertex as a subset of K, and show that this set has property A by induction. It will follow then that the stabilizers in the fundamental group will also have property A.

We proceed by induction. The base case is clear since W_0 is precisely G_{P_0} and by assumption, W_0 has property A. By Proposition 2.6.1, W_R is precisely the set of all words in K with associated path c for which $l(c) \leq R$. Let $H_k \subset K$ be the set of elements whose paths have length precisely k. Then, in the above notation,

$W_R = \cup_{k \leq R} H_k$, so by the Finite Union Theorem (5.4.2), it suffices to prove that H_k has property A for all k. Since W_0 and H_0 coincide, we proceed to the induction step on H_k.

Observe that $H_k \subset \cup_{y \in \text{Edge}Y} H_{k-1} y G_{t(y)}$. Since Y was assumed to be a finite graph, this is a finite union. So, by the Finite Union Theorem, it suffices to show that $H_k \cap H_{k-1} y G_{t(y)}$ has property A, for some fixed $y \in \text{Edge}Y$. Let $Y_r = H_{k-1} y N_r(\phi_y(G_y))$, where $N_r(\phi_y(G_y))$ is the r-neighborhood of $\phi_y(G_y)$ in $F(G, Y)$. Now, the set Y_r is coarsely equivalent to $H_{k-1} y \phi_y(G_y)$, and by the relations on $F(G, Y)$, this set is $H_{k-1} \phi_y(G_y) y = H_{k-1} y$. Finally, $H_{k-1} y$ is coarsely equivalent to H_{k-1}, which, by the inductive hypothesis, has property A. Thus, we conclude that Y_r has property A. Consider the sets $xyG_{t(y)}$, indexed by those $x \in H_{k-1}$, which do not end with an element of $\phi_{\bar{y}}(G_y)$. Notice that this collection does cover $H_k \cap H_{k-1} y G_{t(y)}$ since we may obtain $x\phi_{\bar{y}}(a) y g$ from $x y \phi_y(a) g$, which is of the required form. The map $G_{t(y)} \mapsto xyG_{t(y)}$ is an isometry, so we are in the situation of the infinite union theorem.

To apply the Infinite Union Theorem (5.4.1), we need only show that the family $\{H_k \cap H_{k-1} y G_{t(y)} \setminus Y_r\}$ is r-disjoint. To this end, let $x \neq x'$ and suppose that z and z' are in $G_{t(y)} \setminus N_r(\phi_y(G_y))$. Consider $d(xyz, x'yz') = \|z^{-1} y^{-1} x^{-1} x' y z'\|$. Write $z = \phi_y(a) s$, and $z' = \phi_y(a') s'$, where $\|s\|$ and $\|s'\| > r$. Then,

$$\|z^{-1} y^{-1} x^{-1} x' y z'\| = \|s^{-1} y^{-1} \phi_{\bar{y}}(a^{-1}) x^{-1} x' \phi_{\bar{y}}(a') y s'\|.$$

Now, a reduction can only occur in the middle, and if $\phi_{\bar{y}}(a^{-1}) x^{-1} x' \phi_{\bar{y}}(a')$ is not in $\phi_{\bar{y}}(G_y)$, then

$$\|z^{-1} y^{-1} x^{-1} x' y z'\| > \|s\| + \|s'\| > 2r.$$

So, suppose that there is some $b \in G_y$ so that $\phi_{\bar{y}}(b) = \phi_{\bar{y}}(a^{-1}) x^{-1} x' \phi_{\bar{y}}(a')$. Then, putting $c = ab(a')^{-1}$, we see that $x' = x\phi_{\bar{y}}(c)$. By construction x' cannot end with a nontrivial element of $\phi_{\bar{y}}$, so that $\phi_{\bar{y}}(c) = e$, which implies $x' = x$, a contradiction.

We conclude that the families are r-disjoint, and therefore, by the Infinite Union Theorem, H_k has property A so that W_R has property A for every R. ■

In view of the Bass-Serre structure theorem [16], every group acting without inversion on a tree with compact quotient is a fundamental group of a graph of groups. Combining this with the fact that asdim$T = 1$, for an infinite tree, we obtain as a consequence of Theorem 5.3.1 the following result of Jean-Louis Tu:

Theorem 5.5.2 *Let (G, Y) be a finite graph of groups with finitely generated vertex groups with G_v having property A for all vertex groups. Then for any vertex v_0 the fundamental group $\pi_1(G, Y, v_0)$ has property A.*

In particular, Amalgamated free products and HNN extensions of groups constructed from groups with property A will have property A.

5.6 Complexes of Groups

The analog of Lemma 3.4.1 for Property A is the following. The proof follows from the proof of Lemma 5.5.1.

Lemma 5.6.1 *Let $G(\mathcal{Y})$ be a developable complex of groups. Suppose that $\mathcal{Y}$ a finite, connected SCWOL, that the local groups are finitely generated, and that the local groups have property A. Then, the R-stablizer $W_R(\sigma_0)$ for some base vertex σ_0 also has property A.*

Applying the Lemma, we obtain the following generalization of Tu's theorem.

Theorem 5.6.2 *Let $G(\mathcal{Y})$ be a developable complex of groups with corresponding development $\mathcal{X}$ and fundamental group π. Suppose that* asdim$|\mathcal{X}|$ *is finite and that the stabilizers of the action have property A. Then, π has property A.*

APPENDIX
NOVIKOV AND RELATED CONJECTURES

For completeness, we give the statements of the Novikov conjecture as well as some other closely related conjectures. The purpose of this appendix is to give some idea about the conjectures. It is not intended to be a technical exposition and many of the details will be omitted. More information about these conjectures can be found in Roe [15] and Ferry-Ranicki-Rosenberg [10].

Let Γ be a discrete group. Let $B\Gamma$ denote the classifying space $K(\Gamma, 1)$. An element $x \in H^*(B\Gamma; \mathbb{Q})$ can be interpreted as a characteristic class for manifolds with fundamental group Γ. If M is a manifold with $\pi_1(M) = \Gamma$, then one can find a map $u : M \to B\Gamma$ which induces an isomorphism of the fundamental groups.

In 1971 Hirzebruch proved his famous signature theorem which expresses the signature σ of a closed $4k$-dimensional manifold M in terms of characteristic classes. For oriented manifolds M, the class $x \in H^*(B\Gamma; \mathbb{Q})$ defines a rational number called the *higher signature*, denoted $signature_x(M, u)$. This number is *homotopy invariant* if whenever $f : N \to M$ is an orientation-preserving homotopy equivalence of closed oriented manifolds, and $u : M \to B\Gamma$, then $signature_x(M, U) = signature_x(N, u \circ f)$.

Novikov Conjecture for Γ. The higher signatures determined by Γ are all homotopy invariant.

Borel Conjecture for Γ. Every homotopy equivalence $f : N \to M$ of closed aspherical manifolds with $\pi_1(M) = \Gamma$ is homotopic to a homeomorphism.

It can be shown that the Borel Conjecture for Γ implies the Novikov Conjecture for Γ.

The principle of descent refers to the fact that an analytic version of the Novikov Conjecture follows from the coarse Baum-Connes Conjecture. For completeness, we give the statements of these conjectures. More information including a discussion of the K-theory of C^*-Algebras, the coarse analog of K-homology (KX_*) and assembly can be found in Roe [15].

Coarse Baum-Connes Conjecture If X is of bounded geometry then the coarse assembly map

$$A_\infty : KX_*(X) \to K_*(C^*(X))$$

is an isomorphism.

Analytic Novikov Conjecture Suppose that Γ is a group with $B\Gamma$ a finite complex. Then the assembly map

$$K_*(B\Gamma) \to K_*(C_r^*\Gamma)$$

is injective. (Here $C_r^*\Gamma$ denotes the reduced C^*-Algebra of the group Γ.)

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BIOGRAPHICAL SKETCH

I was born to Robert L. and Laurelle G. Bell in Panama City, Florida, on Saturday, July 24th, 1976. My father was an Air Force officer, so we moved frequently. Shortly after my second birthday, we moved to Naples, Italy. We were then reassigned to Panama City, then to Niceville, Florida, and Tampa, Florida. When my father retired, we settled in Gainesville, Florida. I attended Eastside High School in Gainesville. I earned bachelor's and master's degrees in mathematics at the University of Florida. In December of 2001 I married Santiya Sarntinoranont, whom I had known since seventh grade.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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